HOD-SUPERCOMPACTNESS AND THE OPTIMALITY OF WOODIN'S HOD DICHOTOMY

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Abstract. We answer a question of Goldberg–Poveda from [GOP24] by showing that the HOD hypothesis is consistent with κ being the first HOD-supercompact cardinal and carrying a club of cardinals that are regular in HOD. This configuration yields a sharp distinction between HOD-supercompactness and extendibility when it comes to covering between HOD and the set-theoretic universe. To obtain it, we develop a new forcing construction: a homogeneous Radin-like poset. Finally, we also address a conjecture of Woodin from [Woo10] concerning successors of singular cardinals in HOD.

1. Introduction

Questions of definability have played a central role in set theory since its origins. In a 1946 lecture at Princeton [Göd46], Gödel introduced the class of *hereditarily ordinal definable sets*, nowadays denoted by HOD. Informally, a set X is ordinal definable if it is the unique object satisfying some set-theoretic property that can be described using finitely many ordinal parameters. A set X belongs to HOD if X is ordinal definable, every element of X is ordinal definable, every element of every element of X is ordinal definable, and so on; see §2 for the formal definition. Subsequent work of Myhill–Scott [Myh71], Vopěnka [VH72], and others began to reveal the surprisingly rich structure of the class of ordinal definable sets.

The class HOD is transitive, contains all ordinals, and satisfies the usual axioms of set theory; namely, the *Zermelo–Fraenkel axioms together with the axiom of choice* (ZFC). In modern set-theoretic jargon, HOD is an *inner model* of the set-theoretic universe V. Inner models provide canonical approximations to the universe, and a central theme in set theory is to understand how much of the structure of V they can capture. *Gödel's constructible universe* L ([Göd39]) initiated this program, and subsequent work of Jensen, Mitchell, Steel, Neeman, Woodin, and many others ([Ste09, Zem02]) have produced increasingly sophisticated canonical inner models capable of accommodating stronger and stronger large-cardinal axioms.

The position of HOD within this picture is particularly intriguing. Unlike the fine-structural inner models just mentioned, HOD is defined directly

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from the ambient universe and is highly sensitive to forcing. Nevertheless, it contains every canonical inner model constructed to date and can accommodate virtually all large cardinals (see e.g., [BP23]). Thus the comparison between HOD and V provides a natural testing ground for a basic problem in the foundations of mathematics; namely, *to what extent does the definable part of the universe reflect the structure of the universe itself?*

Large-cardinal axioms postulate the existence of infinite cardinals whose existence cannot be established from ZFC. They form a remarkable hierarchy of principles which calibrates the consistency strength of a broad range of mathematical theories and has played a decisive role in the modern theory of independence [Kan09, Koe11]. Large cardinals have also had consequences for more familiar mathematical objects — for instance, for regularity properties of sets of real numbers [Sol70, DPT25, PT25] and for structural questions in algebra [MS96, EM02, CIP25]. At sufficiently high levels of this hierarchy, their interaction with HOD becomes unexpectedly rigid.

In recent years, Woodin's groundbreaking work [Woo10, Woo17] has stimulated considerable activity around the interaction between HOD, large cardinals, compactness phenomena, and forcing. One direction has sought to understand how dramatically the cardinal and large-cardinal structure of HOD can diverge from that of the ambient universe, using sophisticated forcing techniques; see, among others, [GM18, BNU17, BNH23, GOP24, Pov24]. A complementary line of research has uncovered compactness and covering principles that constrain this divergence in the presence of strongly compact cardinals and singular cardinals of uncountable cofinality; see [Gol24b, GP25, GOP24, BCH$^+$26]. These developments form part of a broader renewed interest in definable inner models and their interaction with large cardinals, also visible in the recent influential work on *Nairian models* [BLS25].

A fundamental discovery of Woodin is the following dichotomy.

Theorem (Woodin's HOD Dichotomy [Woo10])**.** *Suppose that δ is an extendible cardinal. Then exactly one of the following holds:*

(1) HOD *is a weak extender model for the supercompactness of δ;*
(2) every regular cardinal $\lambda \geq \delta$ is ω-strongly measurable in HOD*.*

Let us briefly explain the first alternative, which will be central to this paper. An inner model N is a *weak extender model for the supercompactness of κ* if, for every cardinal $\lambda \geq \kappa$, there is a normal fine κ-complete ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that $\mathcal{P}_\kappa(\lambda) \cap N \in U$ and $U \cap N \in N$. Thus the supercompactness measures of the ambient universe can be chosen so as to concentrate on N while leaving behind measures internal to N. In particular, if N is a weak extender model for the supercompactness of κ, then κ is itself supercompact in N. Additionally, these models enjoy of strong forms of *covering* and correctness properties above κ (see [Woo17]). Among others, there is the *weak covering* property which states the following: if $\lambda > \kappa$ is a singular cardinal then it is singular in N and $\lambda^{+N} = \lambda^+$.

The two alternatives of Woodin's dichotomy therefore describe radically different pictures. In the first, HOD retains a substantial part of the large-cardinal structure of the universe. In the second, HOD is very far from V: all sufficiently large regular cardinal are large cardinals inside HOD.

Woodin's *HOD hypothesis* (see §2) rules out the second alternative. His broader *HOD conjecture* predicts that the HOD hypothesis should follow from suitable large-cardinal assumptions. In this sense, the HOD hypothesis expresses the expectation that, in the presence of sufficiently strong large cardinals, HOD should eventually resemble the universe of sets.

An important point is that ordinary supercompactness does not by itself enforce this relationship with HOD. Woodin introduced the notion of HOD-*supercompactness* (see Definition 2.7) precisely to capture the additional coherence that is required. Every extendible cardinal is HOD-supercompact ([Woo10]), while the consistency strength of HOD-supercompactness is only that of a supercompact cardinal. The distinction between supercompactness and HOD-supercompactness is nevertheless genuine. The first observation in this direction is due to Sargsyan [Sar08]. More recently, Goldberg and Poveda showed that it is consistent for κ to be supercompact and for the HOD hypothesis to hold, while HOD is nevertheless not a weak extender model for the supercompactness of κ. Thus ordinary supercompactness, even together with the HOD hypothesis, does not suffice to recover the first alternative of Woodin's dichotomy. For HOD-supercompact cardinals the situation is completely different: if κ is HOD-supercompact and the HOD hypothesis holds, then HOD is a weak extender model for the supercompactness of κ (Theorem 2.9).

Thus, despite having only the consistency strength of supercompactness, HOD-supercompactness recovers much of the extendible-like rigidity of HOD *above* the large cardinal. This raises the question at the heart of this paper:

Question. *How much extendible-like rigidity of* HOD *does a* HOD*-supercompact cardinal κ impose below itself?*

Recent work of Goldberg–Poveda [GOP24] shows that the situation below an ordinary supercompact cardinal can be remarkably wild. On the one hand, if the HOD hypothesis holds and κ is supercompact, then there are unboundedly many singular cardinals $\lambda < \kappa$ which remain singular in HOD and satisfy $\lambda^{+\mathrm{HOD}} = \lambda^+$.[1] Thus some degree of correctness below κ is unavoidable. On the other hand, this cannot in general be strengthened from "unboundedly many" to "club many". Indeed, it is consistent with the HOD hypothesis that the first supercompact cardinal κ carries a club $C \subseteq \kappa$ such that every $\lambda \in C$ is regular in HOD and $\lambda^{+\mathrm{HOD}} < \lambda^+$.

Question 1 ([GOP24])**.** *Is it consistent with the* HOD *hypothesis that κ is* HOD*-supercompact and there is a club $C \subseteq \kappa$ consisting of* HOD*-regular cardinals θ such that $\theta^{+\mathrm{HOD}} < \theta^+$?*

[1] This degree of resemblance was also observed by Woodin.

The question is particularly natural because the corresponding configuration is incompatible with extendibility, whereas HOD-supercompactness reproduces many of the consequences of extendibility for the structure of HOD. Our main theorem gives a positive partial answer to this question.

Main Theorem. Assume the existence of a supercompact cardinal. Then there is a cardinal-preserving generic extension in which the HOD hypothesis holds and the first HOD-supercompact cardinal κ carries a club $C \subseteq \kappa$ consisting of cardinals which are measurable in HOD.[2]

The proof of the Main Theorem requires a new forcing construction: a *homogeneous Radin-like forcing*. At an intuitive level, our aim is to reproduce one of the characteristic effects of Radin forcing — the introduction of a club of former regular cardinals — while retaining enough homogeneity to control HOD of the final generic extension and enough large-cardinal structure to preserve HOD-supercompactness. The earlier construction from [GOP24], based on Cummings–Friedman–Golshani's work [CFG15], does not seem to yield the configuration described in the Main Theorem. The basic issue is the lack of homogeneity and the effects that it has on HOD-supercompactness.

Our construction achieves the desired configuration by melting two forcing techniques: The first originates in Gitik's classical work on adding closed unbounded sets consisting of former regular cardinals [Git99]; the second is the nonstationary-support iteration of Prikry-type forcings developed by Ben-Neria and Unger [BNU17], which provides the degree of homogeneity needed to control the HOD of a forcing extension. The similarities and differences between our methods and theirs are explained at length in §3.3.

Our expectation is that this novel homogenous Radin-like forcing will find other applications in the study of HOD beyond those pursued here.

Separately, the paper also addresses a conjecture of Woodin concerning successors of singular cardinals in HOD. In [Woo10, p. 299], he observed that if γ is a singular strong limit cardinal and $\gamma^{+\mathrm{HOD}} < \gamma^+$, then either $\mathrm{cf}(\gamma^{+\mathrm{HOD}}) = \omega$ or $\mathrm{cf}(\gamma^{+\mathrm{HOD}}) = \mathrm{cf}(\gamma)$, and conjectured that the first possibility should always occur. We prove this in the negative:

Theorem. *Assume the existence of a κ^+-supercompact cardinal κ with $2^\kappa = \kappa^+$. Then there is a generic extension in which κ is a singular strong limit cardinal, $\mathrm{cf}(\kappa^{+\mathrm{HOD}}) = \omega_1$, and $\kappa^{+\mathrm{HOD}} < \kappa^+$. Moreover, $\mathrm{cf}^{\mathrm{HOD}}(\kappa) = \omega_1$.*

The paper is organized as follows. In Section 2 we review the relevant background on HOD, weak extender models, large cardinals, and iterations of Prikry-type forcings. In Section 3 we define the global iteration and establish its preservation properties. Section 3.1 develops the local forcing notions used in the construction extending Gitik's machinery to the Mitchell orders required for our application. Section 4 contains the homogeneity analysis.

[2] Actually, by starting from a stronger large cardinal assumptions, we can upgrade this to encompass other large cardinal properties in HOD; such as supercompactness in HOD.

In Section 5 we prove the preservation of HOD-supercompactness and establish the Main Theorem. Finally, Section 6 gives the negative answer to Woodin's conjecture concerning successors of singular cardinals in HOD.

2. Preliminaries

Notations and conventions. We follow standard set-theoretic notation. If M is an inner model and λ is a cardinal, we write λ^{+M} for the successor of λ as computed in M. We will also use expressions such as "λ is M-regular" to mean that $M \models$ "λ is regular". When no model is indicated, all cardinal-theoretic notions are computed in the ambient universe.

Given a forcing poset $\mathbb{P}$, we shall adopt the convention that $p \leq q$ means that p *is stronger than* q. We denote the trivial condition of a forcing $\mathbb{P}$ by $\mathbb{1}_{\mathbb{P}}$, or simply by $\mathbb{1}$ when $\mathbb{P}$ is clear from context. We usually suppress the subscript in the forcing relation when the relevant forcing is unambiguous.

Given a forcing poset $\mathbb{P}$, a $\mathbb{P}$-name τ, and a formula $\varphi(x)$ in the forcing language of $\mathbb{P}$, we say that τ *is forced to satisfy* φ whenever $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi(\tau)$. When the context allows it, we will also write $\Vdash \varphi(\tau)$ rather than the more informative $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi(\tau)$.

2.1. **HOD.** The primary object of study in this manuscript is Gödel's universe of Hereditary Ordinal Definable sets, HOD (see [Göd46]):

Definition 2.1. Let X be a set. The class of *Ordinal Definable sets relative to* X, denoted OD_X, consists of all sets A for which there is a formula $\phi(z, x_0, \ldots, x_{m-1}, y_0 \ldots, y_{n-1})$ in the language of set theory, sets $a_0, \ldots, a_{m-1}$ in X and ordinals $\beta, \alpha_0, \ldots, \alpha_{n-1}$ such that A can be presented as

$$A = \{z \in V_\beta \mid V_\beta \models \varphi(z, a_0, \ldots, a_{m-1}, \alpha_0, \ldots, \alpha_{n-1})\}.$$

The class of *Hereditarily Ordinal Definable sets relative to* X, denoted HOD_X, is the class of all A such that $\mathrm{trcl}(\{A\}) \subseteq \mathrm{OD}_X$.

We shall write OD and HOD for $\mathrm{OD}_\emptyset$ and $\mathrm{HOD}_\emptyset$, respectively.

HOD is an *inner model*; namely, a transitive $\in$-structure satisfying the ZFC axioms and containing all the ordinals (through the paper this is what we shall mean when we refer to an *inner model*).

By forcing with a class iteration that codes every set of ordinals into the continuum function, McAloon proved the following well-known theorem:

Fact 2.2 (McAloon, [McA71])**.** *There is a class forcing poset sets yielding a generic extension where the axiom* $V = \mathrm{HOD}$ *holds.*

A strengthening of "$V = \mathrm{HOD}$" relevant to this paper was introduced by Fuchs–Hamkins–Reitz in [FHR15]:

Definition 2.3 (Fuchs–Hamkins–Reitz)**.** The *generic* HOD *of* V, denoted gHOD, is the intersection of the HODs of all the generic extensions of V.

That is, $x \in \mathrm{gHOD}$ if and only if $x \in \mathrm{HOD}^{V[G]}$ for every set forcing $\mathbb{P}$ and $G \subseteq \mathbb{P}$ generic. Equivalently,

$$\mathrm{gHOD} = \{x \mid \forall \mathbb{P}\ \mathbb{1} \Vdash_{\mathbb{P}} \check{x} \in \dot{\mathrm{HOD}}\}$$

Fact 2.4 ([BT09, Theorem 15])**.** *Suppose the* GCH *holds and that κ is supercompact. Then, there is a class iteration forcing $V = \mathrm{gHOD}$ and preserving both the* GCH *and the supercompactness of κ.*

We now present a series of results of Woodin's landmark work [Woo10].

Definition 2.5 (Woodin, [Woo10])**.** Given a cardinal κ, an inner model M of ZFC is a *weak extender model for the supercompactness of κ* (shortly, *WEM for κ*) if for all $X \in M$, there is a normal fine κ-complete ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(X) \cap M$ with $\mathcal{U} \cap M \in M$.

M being a WEM for κ implies that κ is supercompact in both M and V.

WEM possess strong covering properties making them close to V:

Theorem 2.6 (Woodin, [Woo17])**.** *If M is* WEM *for κ then M has the κ-cover and κ-approximation properties. In particular, every singular cardinal $\lambda > \kappa$ is singular in M and $\lambda^{+M} = \lambda$.*

In §5 we shall be concerned with the following notion due to Woodin:

Definition 2.7 (Woodin, [Woo10])**.** κ is HOD-supercompact if for all strong limit cardinals λ, there exists an elementary embedding $j\colon V \to M$ with critical point κ such that ${}^{\lambda}M \subseteq M$ and $\mathrm{HOD}^M \cap V_\lambda = \mathrm{HOD} \cap V_\lambda$.

Every HOD-supercompact cardinal is supercompact, but the converse is false as shown by Sargsyan in [Sar08]. For more results see [AFF21].

Woodin's structural theory of HOD hinges on the following hypothesis:

Definition 2.8 ([Woo17])**.** The HOD *hypothesis* states that there is a proper class of regular cardinals that are not ω-strongly measurable in HOD.

One of his central theorems is the following:

Theorem 2.9 (Woodin [Woo10])**.** *Assume the HOD hypothesis and let κ be* HOD*-supercompact. Then* HOD *is a weak extender model for the supercompactness of κ. In particular, κ is supercompact in* HOD*.*

The above result is actually sharp:

Theorem 2.10 (Goldberg–Poveda, [GOP24])**.** *The following configuration is consistent relative to large cardinals: the* HOD *hypothesis holds, κ is supercompact but not supercompact in* HOD*.*

The HOD hypothesis together with milder form of extendibility, such as supercompactness, already entail some agreement between V and HOD:

Theorem 2.11 (Woodin; Poveda [GOP24])**.** *Assume that κ is a supercompact cardinal and the* HOD *hypothesis holds. Then, for unboundedly many singular cardinals $\theta < \kappa$, θ is singular in* HOD *and $\theta^{+\mathrm{HOD}} = \theta^+$.*[3]

Once again, the above form of agreement is arguably optimal:

Theorem 2.12 (Goldberg–Poveda, [GOP24])**.** *The following configuration is consistent relative to large cardinals: the* HOD *hypothesis holds, κ is supercompact and there is a club $C \subseteq \kappa$ consisting of* HOD*-regular cardinals θ such that $\theta^{+\mathrm{HOD}} < \theta^+$.*

In §5 we shall prove Theorem 5.3 which produces a similar configuration replacing supercompactness by the stronger notion of HOD-supercompactness. This configuration, however, cannot take place below the first extendible, thus evidencing a sharp disctintion between extendibility and HOD-supercompactness when it comes to the structure of HOD as a weak extender model.

Proposition 2.13. *Assume the* HOD *hypothesis and let κ be an extendible cardinal. Then there is some $\kappa_* < \kappa$ such that, for every singular cardinal $\theta \in (\kappa_*, \kappa)$, θ is singular in* HOD *and $\theta^{+\mathrm{HOD}} = \theta^+$.*

Proof. This follows easily from Goldberg's strongly compact version of the HOD dichotomy [Gol24b, Corollary 2.9]. Concretely, if κ_* is the first strongly compact cardinal (which necessarily is less than κ) then the HOD hypothesis implies that every singular cardinal $\theta > \kappa_*$ remains singular in HOD and satisfies $\theta^{+\mathrm{HOD}} = \theta^+$. □

2.2. **Forcing.** In this section we present some forcing preliminaries needed in the main theorem of this paper. Any related concept or notation that has not been defined can be found in Kunen's text [Kun14].

Definition 2.14. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets and $\Gamma\colon \mathbb{P} \to \mathbb{Q}$ any map. We define $\Gamma^*\colon V^{\mathbb{P}} \to V^{\mathbb{Q}}$ by induction on the name-rank, letting

$$\Gamma^*(\tau) := \{\langle \Gamma^*(\sigma), \Gamma(p)\rangle \mid \langle \sigma, p\rangle \in \tau\}$$

Such maps are usually defined in the context of dense embeddings or automorphisms. Typically, homogeneity of posets, which is crucial when dealing with HOD, is defined in terms of the existence of automorphisms. However, in the case of forcing iterations, outright automorphisms are difficult to obtain, since we are dealing with names, so we must weaken the notion somewhat. That this is possible is provided by the following lemma:

Lemma 2.15. *Let $\mathbb{P}$ be a poset and $\Gamma\colon \mathbb{P} \to \mathbb{P}$ an order-preserving map such that for every $p \in \mathbb{P}$, $p \leq \Gamma^2(p) \leq p$. Then Γ is a dense embedding.*

Proof. It remains to show that the image of Γ is dense. However, clearly, for every $p \in \mathbb{P}$, $\Gamma(\Gamma(p)) \leq p$ by assumption. □

[3]This theorem was first observed by Woodin. Poveda observed it independently later while working on [GOP24].

Next, we introduce a slight variation of the usual notion of automorphism of a poset – that will be suitable for our purposes.

Definition 2.16. Let $\mathbb{P}$ be a poset and let $\Gamma\colon \mathbb{P} \to \mathbb{P}$ be a map. We say that Γ is a *self-inverse automorphism modulo equivalence* if Γ preserves the order and for every $p \in \mathbb{P}$, $p \leq \Gamma^2(p) \leq p$.

Lemma 2.17. *Let $\mathbb{P}$ be a poset and $\Gamma\colon \mathbb{P} \to \mathbb{P}$ a self-inverse automorphism modulo equivalence. Let $G \subseteq \mathbb{P}$ be a generic filter. Then $\Gamma^{-1}[G]$ and the upward closure of $\Gamma[G]$ are equal. In particular, the upward closure of $\Gamma[G]$ is a $\mathbb{P}$-generic filter.*

Proof. First suppose that $q \in \Gamma^{-1}[G]$, i.e. $\Gamma(q) \in G$. Then $\Gamma(\Gamma(q)) \leq q$ by assumption and $\Gamma(\Gamma(q)) \in \Gamma[G]$, showing that q belongs to the upward closure of $\Gamma[G]$.

Now suppose that q belongs to the upward closure of $\Gamma[G]$, i.e. $\Gamma(p) \leq q$ for $p \in G$. Then $p \leq \Gamma(\Gamma(p)) \leq \Gamma(q)$, showing that $\Gamma(q) \in G$ and thus $q \in \Gamma^{-1}[G]$.

To see the "in particular" part, note that, since $\Gamma\colon \mathbb{P} \to \mathbb{P}$ is a dense embedding, $\Gamma^{-1}[G]$ is a $\mathbb{P}$-generic filter. □

Lemma 2.18 (Homogeneity)**.** *Suppose that $\mathbb{P}$ is a poset such that for every $p, q \in \mathbb{P}$ there is a dense set $D \subseteq \mathbb{P}$, a condition $p' \leq p$ with $p' \in D$ and a self-inverse automorphism modulo equivalence $\Gamma\colon D \to D$ such that $\Gamma(p')$ and q are compatible. Then whenever $x_0, \dots, x_{n-1}$ are sets, ϕ is a statement in the forcing language and $p \in \mathbb{P}$, we have*

$$p \Vdash \phi(\check{x}_0, \dots, \check{x}_{n-1}) \text{ if and only if } \mathbb{1}_{\mathbb{P}} \Vdash \phi(\check{x}_0, \dots, \check{x}_{n-1})$$

Proof. Clearly, the condition $\mathbb{1}_{\mathbb{P}} \Vdash \phi(\check{x}_0, \dots, \check{x}_{n-1})$ is sufficient, let us show that it is also necessary. To this end, suppose that $p \Vdash \phi(\check{x}_0, \dots, \check{x}_{n-1})$ and assume toward a contradiction that $\mathbb{1}_{\mathbb{P}} \nVdash \phi(\check{x}_0, \dots, \check{x}_{n-1})$, i.e. there is some $q \in \mathbb{P}$ such that $q \Vdash \neg\phi(\check{x}_0, \dots, \check{x}_{n-1})$. By our assumption there is a dense set $D \subseteq \mathbb{P}$, a condition $p' \leq p$ with $p' \in D$ and a self-inverse automorphism modulo equivalence $\Gamma\colon D \to D$ such that $\Gamma(p')$ and q are compatible. First of all, we have

$$p' \Vdash_{\mathbb{P}} \phi(\check{x}_0, \dots, \check{x}_{n-1})$$

Since D is dense, this implies

$$p' \Vdash_D \phi(\check{x}_0, \dots, \check{x}_{n-1})$$

Since Γ is a dense embedding from D to D by Lemma 2.15, this implies that

$$\Gamma(p') \Vdash_D \phi(\check{x}_0, \dots, \check{x}_{n-1})$$

which, again by density of D, implies

$$\Gamma(p') \Vdash_{\mathbb{P}} \phi(\check{x}_0, \dots, \check{x}_{n-1})$$

but this contradicts our assumption of compatibility of $\Gamma(p')$ and q. □

Definition 2.19. Let $\mathbb{P}$ be a forcing poset. We say that $\mathbb{P}$ is *homogeneous modulo equivalence* if for all $p, q \in \mathbb{P}$ there is a dense set $D \subseteq \mathbb{P}$, a condition $p' \leq p$ with $p' \in D$ and a self-inverse automorphism modulo equivalence $\Gamma\colon D \to D$ such that $\Gamma(p')$ and q are compatible.

Using Lemma 2.18 and classical arguments, one sees easily:

Lemma 2.20. *Suppose that $\mathbb{P}$ is an ordinal-definable forcing that is homogeneous modulo equivalence. Then, $\mathbb{1}_{\mathbb{P}} \Vdash \mathrm{HOD}^{V[\dot{G}]} \subseteq \mathrm{HOD}^{V}$.* □

A poset that will be relevant in the main construction of this paper is the forcing shooting a club through a given stationary set S:

Definition 2.21. Let κ be a regular cardinal and $S \subseteq \kappa$ a stationary set. We denote by $\mathbb{CS}(S)$ the forcing whose conditions are closed bounded subsets $c \subseteq \kappa$ with a maximal element $\max(c)$ such that $c \subseteq S$. The order of $\mathbb{CS}(S)$ is reversed end-extension.

Assuming $\kappa^{<\kappa} = \kappa$ it follows that $|\mathbb{CS}(S)| = \kappa$ and so the poset has the κ^+-cc. However, $\mathbb{CS}(S)$ may collapse cardinals less than or equal to κ unless the stationary set S is taken with care. The stationary sets S for which $\mathbb{CS}(S)$ behaves as desired were isolated by Abraham–Shelah in [AS83]:

Definition 2.22. A stationary set $S \subseteq \kappa$ is called *fat* if for every club $C \subseteq \kappa$ the intersection $C \cap S$ contains a closed set of order-type β for all $\beta < \kappa$.

Their main result reads as follows:

Fact 2.23 (Abraham–Shelah)**.** *Let κ be a strongly inaccessible cardinal and $S \subseteq \kappa$ a fat stationary set. Then, $\mathbb{CS}(S)$ is κ-distributive; namely, forcing with $\mathbb{CS}(S)$ does not add new sequences of ordinals of length $<\kappa$.* □.

The main family of posets utilized in this manuscript are of *Prikry-type*:

Definition 2.24. A triple $\langle \mathbb{P}, \leq, \leq^* \rangle$ is called a *Prikry-type forcing* if

(1) $\leq^* \subseteq \leq$,
(2) $\leq^*$ has the *Prikry property*; to wit, for every $p \in P$ and a statement φ in the forcing language of $\langle \mathbb{P}, \leq \rangle$ there is $q \leq^* p$ such that

$$q \Vdash_{\mathbb{P}} \varphi \text{ or } q \Vdash_{\mathbb{P}} \neg\varphi.$$

Any poset is of Prikry-type just letting setting $\leq^* := \leq$. The Prikry-type forcings we shall be interested in will satisfy $\leq^* \subsetneq \leq$ and the direct extension order $\leq^*$ will exhibit better closure properties than the regular order $\leq$.

In [Git86], Gitik introduced Easton-supported iterations of Prikry-type forcings. In this paper, we consider a modification due to Ben-Neria and Unger [BNU17], where the iteration has non-stationary support. This modification allows for finer control over the types of ultrafilters available in the generic extension (see e.g., [HP26b, GK23b, GK23a, HP26a]).

Definition 2.25 (Non-stationary support iteration, [BNU17])**.** Let ϱ be an ordinal. We define an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha \leq \varrho \rangle$ recursively as follows: For $\alpha \leq \varrho$ members of $\mathbb{P}_\alpha$ are sequences $p = \langle \dot{p}_\beta \mid \beta \in \mathrm{supp}(p) \rangle$ such that:

(a) $\mathrm{supp}(p) \subseteq \alpha$ has *non-stationary support*: namely, $\mathrm{supp}(p) \cap \beta$ is *non-stationary* in β for all inaccessible cardinals $\beta \leq \alpha$.

(b) for every $\beta \in \mathrm{supp}(p)$, $p \restriction \beta := \langle \dot{p}_\beta \mid \beta \in \mathrm{supp}(p) \cap \beta \rangle \in \mathbb{P}_\beta$ and

$$p \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{p}_\beta \in \dot{\mathbb{Q}}_\beta.$$

(c) For every $\beta \in \mathrm{supp}(p)$, the weakest condition of $\mathbb{P}_\beta$ forces that:
- $\langle \dot{\mathbb{Q}}_\beta, \leq_\beta, \leq^*_\beta \rangle$ is a Prikry-type forcing notion.
- $|\dot{\mathbb{Q}}_\beta| \leq 2^\beta$.
- There exists a $\leq^*_\alpha$-dense subset $\dot{\mathbb{D}}_\alpha \subseteq \dot{\mathbb{Q}}_\alpha$ such that $\langle \dot{\mathbb{D}}_\alpha, \leq^*_\alpha \rangle$ is α-closed.[4]

Let $p = \langle \dot{p}_\beta \mid \beta \in \mathrm{supp}(p) \rangle$ and $q = \langle \dot{q}_\beta \mid \beta \in \mathrm{supp}(q) \rangle$ be elements of $\mathbb{P}_\alpha$. We write $p \leq_\alpha q$ if and only if the following hold:

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$,
(2) $p \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{p}_\beta \leq_\beta \dot{q}_\beta$, for every $\beta \in \mathrm{supp}(q)$;
(3) there is $b \subseteq \mathrm{supp}(q)$, finite, such that for all $\beta \in \mathrm{supp}(q) \setminus b$

$$p \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{p}_\beta \leq^*_\beta \dot{q}_\beta.$$

The Prikry ordering $\leq^*_\alpha$ is defined by the case in which b above is empty.

Remark 2.26. About (a): While members of $p \in \mathbb{P}_\varrho$ are names for the various posets involved note that $\mathrm{supp}(p)$ is assumed to be a set in the ground model. About (c): the requirement concerning the $\leq^*$-dense sets $\mathbb{D}_\alpha$ replaces the standard assumption that $\langle \mathbb{Q}_\alpha, \leq^*_\alpha \rangle$ is α-closed.

A fundamental theorem regarding the aforementioned iterations is the Prikry property. The result for Easton-supported iterations was proved by Gitik [Git86, §1]; the corresponding one for non-stationary-supported iterations is due to Ben-Neria and Unger [BNU17, §2].

Lemma 2.27 ([BNU17, Lemma 2.2])**.** *For each $\alpha \leq \varrho$, $\langle \mathbb{P}_\alpha, \leq_\alpha, \leq^*_\alpha \rangle$ has the Prikry property.* □

The analysis of nonstationary support iterations heavily relies on the following Fusion Lemma.

Lemma 2.28 (Fusion Lemma, [BNU17, Lemma 2.3])**.** *Suppose that $\alpha \leq \kappa$ is an inaccessible cardinal and $d\colon \alpha \to V$ is a function such that, for each $\beta < \alpha$, $d(\beta)$ is a $\mathbb{P}_{\beta+1}$-name forced by the trivial condition of $\mathbb{P}_{\beta+1}$ to be a dense open subset of $\langle \mathbb{P}_\alpha / \dot{G}_{\beta+1}, \leq^* \rangle$. Then, given $p \in \mathbb{P}_\alpha$ and $\nu < \alpha$ there is $p^* \leq^* p$ and a club $C \subseteq \alpha$ such that $p^* \restriction \nu = p \restriction \nu$ and for each $\beta \in C$, $p^* \restriction \beta + 1 \Vdash \dot{p}^* \setminus \beta + 1 \in d(\beta)$.* □

Hannes: I do not understand this last part

Alejandro: How about now?

[4] Recall that β-closure stands for the following property: For every $\leq^*_\beta$-decreasing sequence $\langle r_\gamma \mid \gamma < \bar{\gamma} \rangle$ of members of $\mathbb{Q}_\beta$ (with $\bar{\gamma} < \beta$) there is $r \in \mathbb{Q}_\beta$ such that $r \leq^*_\beta r_\gamma$.

Lemma 2.29 ([GK24, Lemma 2.5 and Corollary 2.6])**.** *Assume* GCH*. Let α be an inaccessible cardinal and let $\mathbb{P}_\alpha$ be an iteration of Prikry-type forcing satisfying the hypotheses of Definition 2.25.*

(1) Assume that $\dot\eta$ is a $\mathbb{P}_\alpha$-name for an ordinal. Then for every $p \in \mathbb{P}_\alpha$ there exists $q \leq^* p$ and a set $X \in V$, $|X| < \alpha$ such that $q^* \Vdash \dot\eta \in \check X$.
(2) $\mathbb{P}_\alpha$ preserves the inaccessibility of α. □

Corollary 2.30 ([GK24, Theorem 6.5])**.** *Assume* GCH*. Let α be an inaccessible cardinal. Let $\mathbb{P}_\alpha$ be a nonstationary support iteration of Prikry-type forcing satisfying the hypotheses of Definition 2.25. Then $\mathbb{P}_\alpha$ satisfies the α and α^+-cover property.*[5]

Proof. We begin with the α-cover property of $\mathbb{P}_\alpha$. Assume that $A \in V[G]$ is a set of size strictly below α. Let $\dot\eta$ be a $\mathbb{P}_\alpha$-name for an ordinal below α which bounds the cardinality of $\dot A$ in $V[G]$. By Lemma 2.29, there exist $\eta^* < \alpha$ and a condition $p \in G$ such that $p \Vdash |\dot A| < \eta^*$. Work in $V[G \restriction (\eta^*+1)]$ and let

$$\mathbb{R} := \{r \in \mathbb{P}_\alpha \setminus (\eta^*+1) \colon \forall \beta \in \mathrm{supp}(r), r \restriction \beta \Vdash r(\beta) \in \dot{\mathbb{D}}_\beta\}.$$

Using Lemma 2.29 again, build a $\leq^*$-decreasing sequence of conditions $\langle r_i \colon i \leq \eta^*\rangle \subseteq \mathbb{R}$ below $p \setminus (\eta^*+1)$ and sets $\langle X_i \colon i < \eta\rangle$ such that r_i forces that for every $i < \eta^*$, the i-th element of $\dot A$ (if exists) belongs to X_i, and $|X_i| < \alpha$. Note that we have sufficient closure in $\mathbb{R}$ at limit stages, including at the very final stage. By density, we can find a lower bound for such a sequence in $G \setminus (\eta^*+1)$, and this lower bound forces that $\dot A$ is covered by $X^* = \bigcup_{i<\eta^*} X_i$. In particular, there exists a condition $p^* \in G$ such that

$$p^* \restriction (\eta^*+1) \Vdash \exists X^*, |X^*| < \alpha \wedge \left(p^* \setminus (\eta^*+1) \Vdash \dot A \subseteq X^*\right).$$

Since $p^* \in G$ and $\mathbb{P}_{\eta^*+1}$ is small relative to α, there exists $B \in V$ with $|B| < \alpha$ such that $p^* \Vdash \dot A \subseteq B$, as desired.

Next, let us prove the α^+-cover property of $\mathbb{P}_\alpha$. Let $A \in V[G]$ be a set of size α. For simplicity, we identify $\dot A$ with a name for an enumeration of $\dot A$ of order type α. In particular, for every $\beta < \alpha$, let $\dot A(\beta)$ denote the β-th element in the enumeration. For every $\beta < \alpha$, let

$$e(\alpha) = \{r \in \mathbb{P}_\alpha \setminus (\beta+1) \colon \exists X \left(|X| < \alpha \ \wedge r \Vdash \dot A \restriction \beta \subseteq \check X\right)\}.$$

Each set $e(\beta)$ is forced to be a $\leq^*$-dense open subset of $\mathbb{P}_\alpha \setminus (\beta+1)$, by Lemma 2.29 and since $\mathbb{P} \setminus (\beta+1)$ has a $\leq^*$-dense subset which is $(\beta+1)$-closed. By the Fusion Lemma (Lemma 2.28), there exists $p \in G$ and a club $C \subseteq \alpha$ such that, for every $\beta \in C$, there exists a $\mathbb{P}_{\beta+1}$ name for a set of ordinals $\dot X_\beta$ of size strictly below α, such that $p \Vdash \dot A \restriction \beta \subseteq \dot X_\beta$. Since $\mathbb{P}_{\beta+1}$

[5] We say that a poset $\mathbb{P}$ has the κ-cover property if $V \subseteq V[G]$ has the κ-cover property for every V-generic filter $G \subseteq \mathbb{P}$. That is, for every such G and every set of ordinals $A \in V[G]$ with $|A|^{V[G]} < \kappa$, there is a set of ordinals $B \in V$ such that $A \subseteq B$ and $|B|^{V[G]} < \kappa$.

is a small forcing, we can find in V a sequence $\langle X^*_\beta : \beta \in C\rangle$ such that each set X^*_β has size strictly below α, and $p \Vdash \dot{X}_\beta \subseteq \check{X}^*_\beta$. It thus follows that $B := \bigcup_{\beta\in C} X^*_\beta$ is a set in V, $|B| = \alpha$, and $A \subseteq B$. □

Lemma 2.31. *Assume* GCH. *Let α be an inaccessible cardinal. Let $\mathbb{P}_\alpha$ be a nonstationary support iteration of Prikry-type forcing satisfying the hypotheses of Definition 2.25. Let $G \subseteq \mathbb{P}_\alpha$ be generic over V. Then every club in α in $V[G]$ contains a club in α which belongs to V. In particular, forcing with $\mathbb{P}_\alpha$ preserves stationary subsets of α.*

Proof. Let $\dot{C}$ be a $\mathbb{P}_\alpha$-name for a club, and let $\dot{f}$ be a $\mathbb{P}_\alpha$-name for the increasing enumeration of $\dot{C}$. For every $\beta < \alpha$, let $d(\beta)$ be the $\mathbb{P}_{\beta+1}$-name for the set

$$\{q \in \mathbb{P}_\alpha \setminus (\beta+1) \mid \exists \gamma < \alpha (q \Vdash \dot{f}(\beta) < \gamma)\}.$$

Each such set $d(\beta)$ is forced to be a $\leq^*$-dense open subset of $\mathbb{P}_\alpha \setminus (\beta+1)$ by Lemma 2.29. Thus, by the Fusion Lemma (Lemma 2.28), there exists $p \in G$ and a club $D \subseteq \alpha$ such that, for every $\beta \in D$,

$$p \restriction (\beta+1) \Vdash \exists \gamma(\beta) < \alpha \left(p \setminus (\beta+1) \Vdash \dot{f}(\beta) < \gamma(\beta) \right).$$

Let $F : D \to \alpha$ in V be a function such that, for every $\beta \in D$, $p \restriction (\beta+1) \Vdash \dot{\gamma}(\beta) < F(\beta)$ (such F exists since $|\mathbb{P}_{\beta+1}| < \alpha$ for every $\beta < \alpha$). In V, let

$$C^* = \{\beta \in \operatorname{acc}(D) \mid F[D \cap \beta] \subseteq \beta\}.$$

Then $p \Vdash C^* \subseteq \dot{C}$. □

Lemma 2.32 ([BNU17, Corollary 2.6])**.** *Assume* GCH. *Let α be an inaccessible cardinal. Let $\mathbb{P}_\alpha$ be a nonstationary support iteration of Prikry-type forcing satisfying the hypotheses of Definition 2.25. Assume also that for every $\beta < \alpha$, $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta$ preserves cardinals. Then $\mathbb{P}_\alpha$ preserves cardinals.* □

2.3. **Coherent Sequences of Measures.** Our main iteration will be built using Mitchell *coherent sequences of measures* [Mit84]. In this subsection, we will collect the main definitions and prove that, from the consistency of a supercompact cardinal, we can obtain a coherent sequence of measures satisfying our desired properties.

Definition 2.33. Let κ be a cardinal and let $\mathcal{U} = \langle \mathcal{U}_{\alpha,\beta} \mid \alpha \leq \kappa, \beta < o^{\mathcal{U}}(\alpha)\rangle$ be a sequence. We say that $\mathcal{U}$ is a *coherent sequence of measures* if the following holds:

(1) For every $\alpha \leq \kappa$ and every $\beta < o^{\mathcal{U}}(\alpha)$, $\mathcal{U}_{\alpha,\beta}$ is a measure over α;[6]
(2) For every $\alpha \leq \kappa$ and every $\beta < o^{\mathcal{U}}(\alpha)$, letting $j_{\mathcal{U}_{\alpha,\beta}}$ be the ultrapower embedding associated to $\mathcal{U}_{\alpha,\beta}$, we have

$$j_{\mathcal{U}_{\alpha,\beta}}(\mathcal{U}) \restriction (\alpha+1, 0) = \mathcal{U} \restriction (\alpha, \beta)$$

In more detail, point (2) of the above definition states that the following holds in $M_{\mathcal{U}_{\alpha,\beta}}$:

[6] I.e., an α-complete normal ultrafilter on α.

(1) $o^{j_{\mathcal{U}_{\alpha,\beta}}(\mathcal{U})}(\alpha) = \beta$;
(2) For every $\gamma < \beta$, $j_{\mathcal{U}_{\alpha,\beta}}(\mathcal{U})_{\alpha,\gamma} = \mathcal{U}_{\alpha,\gamma}$.

To simplify the notation, we will denote $j_{\mathcal{U}_{\alpha,\beta}}$ simply by $j_{\alpha,\beta}$ and $M_{\mathcal{U}_{\alpha,\beta}}$ by $M_{\alpha,\beta}$ if the coherent sequence $\mathcal{U}$ is clear from context (from Section 3 onwards, we will fix a single coherent sequence).

Definition 2.34. Let κ be a cardinal and let $\mathcal{U} = \langle \mathcal{U}_{\alpha,\beta} \mid \alpha \leq \kappa, \beta < o^{\mathcal{U}}(\alpha) \rangle$ be a coherent sequence. We let $\mathrm{dom}_1(\mathcal{U})$ consist of all α such that $o^{\mathcal{U}}(\alpha) \neq 0$.

We will now show that, from a supercompact cardinal, we can derive a nice coherent sequence of measures:

Lemma 2.35. *Let κ be a supercompact cardinal and assume $V = \mathrm{HOD}$. There exists a coherent sequence of measures*

$$\mathcal{U} = \langle \mathcal{U}_{\alpha,\beta} \colon \alpha \leq \kappa, \beta < o^{\mathcal{U}}(\alpha) \rangle$$

with the following properties:

(1) For every $\alpha \leq \kappa$, $o^{\mathcal{U}}(\alpha) \leq \alpha^+$, and $o^{\mathcal{U}}(\kappa) = \kappa^+$.
(2) Each measure $\mathcal{U}_{\alpha,\beta}$ with $o^{\mathcal{U}}(\alpha) \neq 0$ concentrates on the set of measurable cardinals $\alpha^* < \kappa$ such that $\ell''\alpha^* \subseteq V_{\alpha^*}$.
(3) There are arbitrarily large cardinals $\lambda > \kappa$ such that every fine, normal measure $\mathcal{V}$ on $\mathcal{P}_\kappa(\lambda)$ satisfies $j_{\mathcal{V}}(\mathcal{U}) \restriction \kappa + 1 = \mathcal{U}$.

Proof. Since $V = \mathrm{HOD}$, we can fix an ordinal definable well order $\prec$ of V, which is induced by the natural lexicographic order on finite sequences of ordinals[7] such that for sufficiently large strong limit cardinals θ, $\prec \cap (V_\theta \times V_\theta)$ has order type θ. Let us construct $\mathcal{U}$ by induction as follows. Assume $\alpha \leq \kappa$ is measurable and $\mathcal{U} \restriction (\alpha, \beta)$ has been constructed for some ordinal $\beta \leq \alpha^+$. If $\beta = \alpha^+$, let $o^{\mathcal{U}}(\alpha) = \alpha^+$. Else, let $\mathcal{U}_{\alpha,\beta}$ be the $\prec$-least normal measure W on α which concentrates on measurable cardinals $\alpha^* < \alpha$ with $\ell''\alpha^* \subseteq V_{\alpha^*}$, and satisfies

$$j_W(\mathcal{U} \restriction \alpha)(\alpha) = \langle U(\alpha, \gamma) \colon \gamma < \beta \rangle,$$

if such W exists. If such W does not exist, we let $o^{\mathcal{U}}(\alpha) = \beta$.

Hannes: Induction on what?

Eyal: On elements in $\mathrm{dom}(\mathcal{U})$, namely pairs of ordinals. Do you think that something should be added here to explain this?

Note that $o^{\mathcal{U}}(\alpha) \leq \alpha^+$ for every $\alpha < \kappa$.

It remains to prove that $\mathcal{U}$ is as desired. It is not hard to verify that $\mathcal{U}$ is coherent. Let us claim that $o^{\mathcal{U}}(\kappa) = \kappa^+$. Assume otherwise. Fix any beth-fixed point $\lambda > 2^{2^\kappa}$ and a supercompactness measure $\mathcal{V}$ on $\mathcal{P}_\kappa(\lambda)$. Let $j_{\mathcal{V}} \colon V \to M$ be the associated elementary embedding, with critical point κ, such that ${}^\lambda M \subseteq M$. In particular, since $(V_\lambda)^M = V_\lambda$ and by the definition of $\prec$, we obtain that $j_{\mathcal{V}}(\prec) \restriction V_\lambda = \prec \restriction V_\lambda$. In particular, the measures $\langle \mathcal{U}_{\kappa,\beta} \colon \beta < o^{\mathcal{U}}(\kappa) \rangle$ are chosen in the exact same way over M, and are definable there. Next, let $W = \{X \subseteq \kappa \colon \kappa \in j_{\mathcal{V}}(X)\}$ be the normal measure derived from $\mathcal{V}$. Note that W concentrates on $\{\alpha^* < \kappa \colon \ell''\alpha^* \subseteq V_{\alpha^*}\}$ since $j_{\mathcal{V}}(\ell) \restriction \kappa = \ell$. Let

Hannes: κ^+, no?

Alejandro: Fixed.

Hannes: $j_{\mathcal{V}}$ here?

Alejandro: Fixed

[7] Namely, for a set A, let θ_A be the least beth fixed-point such that A is ordinal definable in V_{θ_A}, and let $\langle n^A, \alpha_0^A, \ldots, \alpha_{m_A}^A \rangle \in V_{\theta_A}$ be the lexicographically least finite sequence for which $n^A \in \mathbb{N}$ codes a formula φ that defines A in V_{θ_A} from the ordinals $\langle \alpha_0^A, \ldots, \alpha_{m_A}^A \rangle$. We then order $A < B$ if $\langle \theta^A, n^A, \alpha_0^A, \ldots, \alpha_{m_A}^A \rangle <_{\mathrm{lex}} \langle \theta^B, n^B, \alpha_0^B, \ldots, \alpha_{m_B}^B \rangle$.

$k\colon M_W \to M_{\mathcal{V}}$ be the standard embedding mapping $[f]_W$ to $j_{\mathcal{V}}(f)(\kappa)$. We have that $\mathrm{crit}(k) = (\kappa^{++})^{M_W}$. Furthermore, again by definability, we have that $\langle \mathcal{U}_{\kappa,\beta} \colon \beta < o^{\mathcal{U}}(\kappa)\rangle \in \mathrm{range}(k)$. In particular, since $\mathrm{crit}(k) > \kappa^+$, we have $\langle \mathcal{U}_{\kappa,\beta} \colon \beta < o^{\mathcal{U}}(\kappa)\rangle \in M_W$, and $j_W(\mathcal{U}) \restriction \kappa + 1 = \mathcal{U}$. Thus we could not have had $o^{\mathcal{U}}(\kappa) < \kappa^+$, since otherwise W was eligible for the role of $\mathcal{U}_{\kappa,o^{\mathcal{U}}(\kappa)}$.

The above proof also establishes that $j_{\mathcal{V}}(\mathcal{U}) \restriction \kappa+1 = \mathcal{U}$, again by the facts that $(V_\lambda)^M = V_\lambda$ and the orders $j_{\mathcal{V}}(\prec)$ and $\prec$ agree on V_λ. □

Remark 2.36. From the proof of Lemma 2.35, it is easy to see that there is a lot of flexibility regarding the choice of what the measures should concentrate on. The only requirement is that this should hold for κ in the target model of the chosen λ-supercompact embedding.[8] Therefore, with the same large cardinal assumption, we could have that the measures concentrate on cardinals with an arbitrary (but bounded) degree of supercompactness. With a stronger large cardinal assumption, we may even have the measures concentrate on fully supercompact cardinals.

3. The main iteration

Hannes: Do we want numbers on this?
Alejandro: I guess they are convenient for reference purposes.

The following are the setup assumptions for the rest of this section:

Setup 3. Assume GCH holds, $V = g\mathrm{HOD}$, κ is a supercompact cardinal, $\ell\colon \kappa \to V_\kappa$ is a Laver function for κ, and

$$\mathcal{U} = \langle U_{\alpha,\tau} \mid \alpha \leq \kappa \wedge \tau < o^{\mathcal{U}}(\alpha)\rangle$$

is a coherent sequence of normal measures on cardinals $\alpha \leq \kappa$ satisfying the hypotheses of Lemma 2.35.

Our main forcing $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ will be a $\mathrm{dom}_1(\mathcal{U})$-spaced nonstationary support iteration of Prikry type forcings of length $(\kappa+1)$. The construction of the iteration is by induction on $\alpha \leq \kappa$. Let us describe the forcings used at each stage $\alpha \leq \kappa$. The picture is the following:

Definition 3.1 (The iteration)**.** We shall denote by $\mathbb{P}_{\kappa+1}$ the $\mathrm{dom}_1(\mathcal{U})$-spaced nonstationary support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha \leq \kappa\rangle$ which for each $\alpha \leq \kappa$ is defined as follows:

- If $\alpha \notin \mathrm{dom}_1(\mathcal{U})$ then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is the trivial forcing".
- If $\alpha \in \mathrm{dom}_1(\mathcal{U})$ but $o^{\mathcal{U}}(\alpha) < \alpha^+$ then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{Q}_\alpha^{o^{\mathcal{U}}(\alpha)}$" (see §3.1).
- If $\alpha \in \mathrm{dom}_1(\mathcal{U})$ and $o^{\mathcal{U}}(\alpha) = \alpha^+$ then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}, \leq^*_{\mathbb{R}_\alpha}\rangle$".
- If $\alpha = \kappa$ then $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}, \leq^*_{\mathbb{R}_\alpha}\rangle$".

[8] This hypothesis can be realized as follows: Let κ be a $C^{(2)}$-extendible cardinal [Bag12]. By a result of Bagaria–Goldberg [BG24], for every $\lambda > \kappa$ with $V_\lambda \prec_{\Sigma_2} V$ there is a λ-supercompact embedding $j\colon V \to M$ with $M \models$ "$V_\lambda \prec_{\Sigma_2} V$". Since κ is supercompact and this is a Π_2-definable property, V_λ thinks that κ is supercompact and then, since λ is Σ_2-elementary inside M, the same is thought by M as well.

If $o^{\mathcal{U}}(\alpha) < \alpha^+$ then we force with the adaptation of Gitik's poset $\mathbb{Q}_\alpha^{o^{\mathcal{U}}(\alpha)}$ from [Git86, §3] presented in §3.1. In case $o^{\mathcal{U}}(\alpha) = \alpha^+$, $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}, \leq^*_{\mathbb{R}_\alpha} \rangle$ is going to be a Prikry-type poset such that $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha} \rangle$ is forcing equivalent to $\mathbb{C}_\alpha$[9] and $\langle \mathbb{R}_\alpha, \leq^*_{\mathbb{R}_\alpha} \rangle$ has a $\leq^*_{\mathbb{R}_\alpha}$-dense subset which is α-closed (see Lemma 3.26). We introduce $\mathbb{R}_\alpha$ in Subsection 3.2.

3.1. **The basic building block.** We now define a variant of Gitik's forcing $\mathbb{P}(\alpha, o^{\mathcal{U}}(\alpha))$ that works for measurable cardinals α such that $o^{\mathcal{U}}(\alpha) \leq \alpha^+$. In [Git86], Gitik assumes that $o^{\mathcal{U}}(\alpha) \leq \alpha$ for all $\alpha \leq \kappa$ but we shall show that his construction can be naturally modified to encompass measurables with larger Mitchell order. Our assumptions hereafter are as in Setup 3.

Fix $\alpha \leq \kappa$. Assume that $\alpha \in \mathrm{dom}_1(\mathcal{U})$, $\mathbb{P}_\alpha$ has been defined, and $G_\alpha \subseteq \mathbb{P}_\alpha$ is generic over V. Assume that for every $\beta \in \mathrm{dom}_1(\mathcal{U}) \cap \alpha$ with $o^{\mathcal{U}}(\beta) < \beta^+$, a set $b_\beta \subseteq \beta$ has been added in $V[G_\alpha]$, such that $b_\beta \subseteq \beta$ is a closed unbounded subset of β, and

$$\mathrm{otp}(b_\beta) = \begin{cases} \omega^{o^{\mathcal{U}}(\beta)} & \text{If } o^{\mathcal{U}}(\beta) < \beta \\ \beta & \text{If } o^{\mathcal{U}}(\beta) \geq \beta \end{cases}.$$

For each $\tau \in (\alpha, \alpha^+)$ we denote by $f_\tau \colon \alpha \to \alpha$ the *canonical function for* τ: For $\tau < \alpha$, f_τ will denote the constant function with value τ and for $\tau = \alpha$ we set $f_\tau := \mathrm{id}$. Note that by canonicity of these maps, $\tau = j_{\mathcal{W}}(f_\tau)(\alpha)$ for all normal measures $\mathcal{W}$ on α. We will only use the notation f_τ when α is clear from the context.

For the rest of the section we we work in $V[G_\alpha]$.

Definition 3.2. Let $\tau \leq o^{\mathcal{U}}(\alpha)$. A sequence $\langle \delta_0, \ldots, \delta_{n-1} \rangle$ in $[\alpha]^{<\omega}$ is called τ-coherent if it is increasing and the following hold:

(1) $o^{\mathcal{U}}(\delta_i) < f_\tau(\delta_i)$ for all $i < n$;
(2) For each $i < n$ let $n_i < i$ be the greatest index such that for each $j \in [n_i, i)$, $o^{\mathcal{U}}(\delta_j) < o^{\mathcal{U}}(\delta_i)$.[10] Then, $\bigcup_{j \in [n_i, i)} b_{\delta_j} \cup \{\delta_j\} \sqsubseteq b_{\delta_i}$.

Definition 3.3. For each τ-coherent sequence t and $\sigma < \tau$ we define $\mathcal{U}_{\alpha,\sigma}(t)$ as the collection of those $X \in \mathcal{P}^{V[G_\alpha]}(\alpha)$ for which there are $p \in G_\alpha$ and a $\mathbb{P}_\alpha$-name $\dot{T}$ such that $p \Vdash_{\mathbb{P}_\alpha} \langle t \restriction \sigma, \dot{T} \rangle \in \mathbb{Q}^\alpha_\sigma$ such that

$$p^\frown \langle t \restriction \sigma, \dot{T} \rangle^\frown j_{\alpha,\sigma}(p) \setminus (\alpha + 1) \Vdash_{j_{\alpha,\sigma}(\mathbb{P}_\alpha)} \alpha \in j_{\alpha,\sigma}(\dot{X}).$$

Lemma 3.4. *Let $\sigma < \tau \leq o^{\mathcal{U}}(\alpha)$ and t a τ-coherent sequence (in $V[G_\alpha]$). Then, the following hold:*

(1) $\mathcal{U}_{\alpha,\sigma}(t)$ is an α-complete ultrafilter in $V[G_\alpha]$ extending $\mathcal{U}_{\alpha,\sigma}$.
(2) $\{\nu < \alpha \mid f_\sigma(\delta) = o^{\mathcal{U}}(\delta)\} \in \mathcal{U}_{\alpha,\sigma}(t)$.

[9] In particular, forcing with $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha} \rangle$ will add a club subset of α through the set of V-regular cardinals. The reason we use $\mathbb{C}_\alpha$, the α-support product of α^+ copies of club shooting $\mathbb{CS}_\alpha$, rather than a single copy of club shooting forcing, is related to the verification of homogeneity of $\mathbb{P}_\kappa$. See Remark 3.27.

[10] Note that n_i could be i, in which case the set of those j's is empty.

(3) $\{\nu < \alpha \mid t \restriction \sigma^\frown\langle\nu\rangle$ *is* τ*-coherent*$\} \in \mathcal{U}_{\alpha,\sigma}(t)$.

Proof. (1). This follows from [BNU17, Proposition 3.3].

(2). This is because $\{\nu < \alpha \mid f_\sigma(\delta) = o^{\mathcal{U}}(\delta)\} \in \mathcal{U}_{\alpha,\sigma}$.

(3). Here is where the canonicity of the function f_τ plays a key role. Let $\dot{X}$ be a $\mathbb{P}_\alpha$-name for the above displayed set and $p \in G_\alpha$ forcing that this name consist of all $\nu < \alpha$ such that $t \restriction \sigma^\frown\langle\nu\rangle$ is τ-coherent. Then, $p^\frown\langle t \restriction \sigma, [\kappa \setminus \max(t \restriction \sigma) + 1]^{<\omega}\rangle$ forces that $\dot{b}_{t\restriction\sigma} \sqsubseteq \dot{b}_\alpha$, ergo it forces that the concatenation $t \restriction \sigma^\frown\langle\alpha\rangle$ satisfies Clause (2) of Definition 3.2. Note that this sequence also witnesses Clause (3): For each $\delta \in \mathrm{range}(t \restriction \sigma)$, $o^{j_{\alpha,\sigma}(\mathcal{U})}(\delta) = o^{\mathcal{U}}(\delta) < f_\tau(\delta) = j_{\alpha,\sigma}(f_\tau)(\delta)$, by coherency of $\mathcal{U}$. Similarly, $o^{j_{\alpha,\sigma}(\mathcal{U})}(\alpha) = \sigma < \tau = j_{\alpha,\sigma}(f_\tau)(\alpha)$ by coherency of $\mathcal{U}$ and by canonicity of f_τ.

In particular, $p^\frown\langle t \restriction \sigma, [\kappa \setminus \max(t \restriction \sigma) + 1]^{<\omega}\rangle^\frown j_{\alpha,\sigma}(p) \setminus \alpha + 1$ forces $\alpha \in j_{\alpha,\sigma}(\dot{X})$, which implies that $\dot{X}^{G_\alpha} \in \mathcal{U}_{\alpha,\sigma}(t)$. □

Definition 3.5. Conditions in $\mathbb{Q}^\tau_\alpha$ consists of pairs $\langle t, T\rangle$ such that t is a τ-coherent sequence and T is a tree on $[\alpha]^{<\omega}$, with root $\varnothing$, such that:

- Every $\nu \in \mathrm{Suc}_T(\varnothing)$ is such that $\max(t) < \nu$.
- For each $\vec{\nu} \in T$, $\mathrm{Suc}_T(\vec{\nu})$ is a union of sets $\mathrm{Suc}_{T,\sigma}(\vec{\nu}) \in \mathcal{U}_{\alpha,\sigma}(t^\frown\vec{\nu})$, for $\sigma < \tau$, such that $\mathrm{Suc}_{T,\sigma}(\vec{\nu}) \subseteq \{\nu < \alpha \mid t^\frown\vec{\nu}^\frown\langle\nu\rangle$ is τ-coherent$\}$.

We say that $\langle t, T\rangle, \langle s, S\rangle$ are *equivalent* if $b_t = b_s$ and $S = T$.

Given $\langle t, T\rangle, \langle s, S\rangle \in \mathbb{Q}^\tau_\alpha$ we write $\langle t, T\rangle \leq^* \langle s, S\rangle$ if $b_t = b_s$ and $T \subseteq S$.

Hannes: Added Something

Remark 3.6 (On our $\leq^*$-order). Note that our pure extension ordering differs from Ben-Neria–Unger: In [BNU17], $\langle t, T\rangle \leq^* \langle s, S\rangle$ if $t = s$ and $T \subseteq S$, so our pure extension ordering is weaker. This is done so that, in the proof of homogeneity, we may work with particular stems.

Alejandro: What is particular stems?

However, this change is a non-issue: Clearly, if $\leq_0 \subseteq \leq_1 \subseteq \leq$ and $\langle \mathbb{P}, \leq, \leq_0\rangle$ has the Prikry property, so does $\langle \mathbb{P}, \leq, \leq_1\rangle$. Therefore, the only potential problem from relaxing the pure extension ordering is losing the closure of $\langle \mathbb{Q}^\tau_\alpha, \leq^*\rangle$, since all other properties we need just depend on $\langle \mathbb{Q}^\tau_\alpha, \leq\rangle$.

Definition 3.7 (One-point extensions)**.** Given $\langle t, T\rangle \in \mathbb{Q}^\tau_\alpha$ and $\langle\nu\rangle \in T$ we denote by $\langle t, T\rangle^\frown\langle\nu\rangle$ the pair $\langle t^\frown\langle\nu\rangle, T_{\langle\nu\rangle}\rangle$ where $T_{\langle\nu\rangle}$ denotes the tree T "above ν"; to wit, $\{\vec{\eta} \in [\alpha]^{<\omega} \mid \langle\nu\rangle^\frown\vec{\eta} \in T\}$. More generally, for $\vec{\nu} \in T$, $\langle t, T\rangle^\frown\vec{\nu}$ is defined by recursion in the obvious fashion.

Definition 3.8. Given $\langle t, T\rangle, \langle s, S\rangle \in \mathbb{Q}^\tau_\alpha$ we write $\langle t, T\rangle \leq \langle s, S\rangle$ if there is $\vec{\nu} \in S$ such that $\langle t, T\rangle \leq^* \langle s, S\rangle^\frown\vec{\nu}$.

Remark 3.9. Note that if $\langle t, T\rangle, \langle s, S\rangle$ are equivalent then $\langle t, T\rangle \leq \langle s, S\rangle$ and $\langle s, S\rangle \leq \langle t, T\rangle$. In particular, they force the same information. This implies that, whenever t and s are equivalent τ-coherent sequences and $\sigma < o^{\mathcal{U}}(\alpha)$, $\mathcal{U}_{\alpha,\sigma}(t) = \mathcal{U}_{\alpha,\sigma}(s)$ (for details, see [Git86]).

Alejandro: in fact, $\leq^*$, right?

Hannes: Added Something.

The following lemma collects the basic properties of $\mathbb{Q}^\tau_\alpha$:

Lemma 3.10 (Basic Properties)**.**

(1) $\langle \mathbb{Q}^\tau_\alpha, \leq \rangle$ *is* α^+*-cc;*
(2) $\langle \mathbb{Q}^\tau_\alpha, \leq^* \rangle$ *is* α*-directed-closed;*
(3) $\langle \mathbb{Q}^\tau_\alpha, \leq \rangle$ *adds a club set* $b_\alpha \subseteq \alpha$ *with* $\mathrm{otp}(b_\alpha) = \omega^\tau$ *consisting of* V_0*-measurable cardinals.* □

Hannes: Added this.

Proof. For (1) and (3), see e.g. [HP26a, Lemma 5.31].

To see (2), let $\{\langle t_i, T_i\rangle \mid i < j\}$ for $j < \alpha$ be a $\leq^*$-directed set. In particular, for every $i, i' < j$, t_i and $t_{i'}$ are equivalent. This implies that the trees T_i split according to the same measures (see Remark 3.9), so, letting $T^* := \bigcap_{i<j} T_i$ we obtain a legitimate tree. Thus, $\langle t_0, T^*\rangle$ is a lower bound of the sequence $\{\langle t_i, T_i\rangle \mid i < j\}$. □

Let us now tackle the usual Prikry-type analysis of the poset.

Definition 3.11. Fix $\tau \leq o^{\mathcal{U}}(\alpha)$. Given a condition $p = \langle t, T\rangle$ in $\mathbb{Q}^\tau_\alpha$, a tree $\Upsilon \in [\alpha]^{\leq n}$ (with $n < \omega$) is called a *p-fat tree (of height n)* if Υ consists of increasing sequences, $\Upsilon \subseteq T$, and for each $\vec{\nu} \in [\Upsilon]^{\leq m}$ with $m < n$, there is an ordinal $\tau(\vec{\nu}) < \tau$ such that $\mathrm{Suc}_\Upsilon(\vec{\nu}) \in \mathcal{U}_{\alpha,\tau(\vec{\nu})}(t^\frown\vec{\nu})$.

We shall denote by $[\Upsilon]$ the set of maximal branches in Υ.

Remark 3.12. Let $D \subseteq \mathbb{Q}^\tau_\alpha$ be a dense open set. Notice that D is closed under equivalence; to wit, if $p \in D$ and $q \in \mathbb{Q}^\tau_\alpha$ is equivalent to p then $q \in D$. To show this, let $D^* := \{q \in \mathbb{P} \mid \exists p \in D\,(p \text{ is equivalent to } q)\}$. Clearly, $D \subseteq D^*$ and D^* is closed under equivalences. Given any $q \in D^*$ there is $p \in D$ that is equivalent to it, hence $q \leq p$ and by opennes of D, $q \in D$. This shows that $D = D^*$.

Hannes: Does this not just follow from $q \leq p$ by equivalence?

Alejandro: I think so. Isn't this what we say?

Lemma 3.13 (Strong Prikry Property)**.** *Fix* $\tau \leq o^{\mathcal{U}}(\alpha)$*. Let* $p \in \mathbb{Q}^\tau_\alpha$ *and* $D \subseteq \mathbb{Q}^\tau_\alpha$ *be a dense open set. Then, there is* $p^* \leq^* p$ *and a* p^**-fat tree* Υ *such that* $p^{*\frown}\vec{\nu} \in D$ *for all* $\vec{\nu} \in [\Upsilon]$*.*

Proof. Let $p = \langle t, T\rangle$ and $D \subseteq \mathbb{Q}^\tau_\alpha$ be as in the lemma.

If there is $p^* \leq^* p$ in D then we choose such a condition and let $\Upsilon := \varnothing$ and we are done. So, suppose that no $\leq^*$-extension of p enters the dense set D.

For each $\tau < o^{\mathcal{U}}(\alpha)$ let us consider

$$X^0_\tau := \{\delta \in \mathrm{Suc}_{T,\tau}(\varnothing) \mid \exists S\,(\langle t^\frown\delta, S\rangle \leq^* p^\frown\langle\delta\rangle \text{ and } \langle t^\frown\delta, S\rangle \in D)\}$$

and let X^1_τ be the complement of X^0_τ.

For each $\tau < o^{\mathcal{U}}(\alpha)$ there is a unique $i_\tau \in \{0,1\}$ such that $X^{i_\tau}_\tau \in \mathcal{U}_{\alpha,\tau}(t)$.

Let T_1 be the tree whose first level is $X = \bigcup_{\tau<o^{\mathcal{U}}(\alpha)} X^{i_\tau}_\tau$ and for each $\delta \in X$ if there is some $\tau < o^{\mathcal{U}}(\alpha)$ such that $i_\tau = 0$ then $(T_1)_{\langle\delta\rangle}$ is some tree $S(\delta)$ that witnesses it; otherwise, stipulate that $(T_1)_{\langle\delta\rangle} = T_{\langle\delta\rangle}$.

Set $p_1 := \langle t, T_1\rangle$. Note that $p_1 \leq^* p$, ergo there is no $q \leq^* p_1$ in D. However, if there is a one-step-extension $q \leq p_1$ in D then there is a p_1-fat tree Υ of height 1 such that $p_1{}^\frown\langle\nu\rangle \in D$ for all $\langle\nu\rangle \in \Upsilon$. Indeed, if there is such a q then $q \leq^* p_1{}^\frown\langle\delta\rangle$ for some $\delta \in X^{i_\tau}_\tau$. Note that $i_\tau = 0$ because the

tree T^q witnesses that $\delta \in X^0_\tau$. Ergo, for every $\delta' \in X^0_\tau$,

$$p_1{}^\frown\langle\delta'\rangle = \langle t^\frown\langle\delta'\rangle, (T_1)_{\langle\delta'\rangle}\rangle = \langle t^\frown\langle\delta'\rangle, S(\delta')\rangle \in D.$$

In this scenario, choosing $p^* = p_1$ and $\Upsilon = X^0_\tau$ we are done.

Suppose that no one-point extension of p_1 enters D. In that case we continue by induction, thus producing a $\leq^*$-decreasing sequence of conditions $\langle p_n \mid n < \omega\rangle$, $\leq^*$-below p, such that the following hold:

$\exists q\,(q \leq^n p_n$ and $q \in D \Rightarrow \exists\Upsilon(\Upsilon$ is a p_n-fat tree and $p_n{}^\frown\vec{\nu} \in D$ for $\vec{\nu} \in [\Upsilon])$.

Let us spell out the details of the construction for the case $n = 2$.

Suppose that p_1 is given. For each $\nu \in \mathrm{Lev}_1(T_1)$ and $\tau < o^{\mathcal{U}}(\alpha)$ let

$$X^0_\tau(\nu) := \{\delta \in \mathrm{Suc}_{T,\tau}(\langle\nu\rangle) \mid \exists S\,(\langle t^\frown\langle\nu,\delta\rangle, S\rangle \leq^* p_1{}^\frown\langle\nu,\delta\rangle \text{ and } \langle t^\frown\langle\nu,\delta\rangle, S\rangle \in D)\}$$

and $X^1_\tau(\nu)$ be the complement of $X^0_\tau(\nu)$.

For each $\nu \in \mathrm{Lev}_1(T_1)$ and $\tau < o^{\mathcal{U}}(\alpha)$ let $i_{\nu,\tau} \in \{0,1\}$ be the unique index such that $X^{i_\tau}_\tau(\nu) \in \mathcal{U}_{\alpha,\tau}(t^\frown\langle\nu\rangle)$. Set $X(\nu) := \bigcup_{\tau<o^{\mathcal{U}}(\alpha)} X^{i_\tau}_\tau(\nu)$. Clearly, this belongs to the intersection of the ultrafilters $\mathcal{U}_{\alpha,\tau}(t^\frown\langle\nu\rangle)$.

Now, for each $\tau,\sigma < o^{\mathcal{U}}(\alpha)$ and $j \in \{0,1\}$, look at

$$Y^{\sigma,j}_\tau := \{\nu \in \mathrm{Succ}_{T_1,\sigma}(\varnothing) \mid X^j_\tau(\nu) = X^{i_{\nu,\tau}}_\tau(\nu)\}.$$

For each τ,σ there is a unique $j_{\tau,\sigma}$ such that $Y^{\sigma,j_{\sigma,\tau}}_\tau \in \mathcal{U}_{\alpha,\sigma}(t)$.

Let $Y^\sigma := \bigcup_{\tau<o^{\mathcal{U}}(\alpha)} Y^{\sigma,j_{\sigma,\tau}}_\tau$. Clearly, this belongs to $\mathcal{U}_{\alpha,\sigma}(t)$.

Let T_2 be the tree whose first level is $\bigcup_{\sigma<o^{\mathcal{U}}(\alpha)} Y^\sigma$.

For each $\nu \in \mathrm{Lev}_1(T_2)$ let

$$\mathrm{Suc}_{T_2}(\langle\nu\rangle) := \bigcup_{\{\sigma\mid\nu\in Y^\sigma\}} \bigcup_{\tau<o^{\mathcal{U}}(\alpha)} X^{j_{\sigma,\tau}}_\tau(\nu).$$

This belongs to the intersection of the filters $\mathcal{U}_{\alpha,\tau}(t^\frown\langle\nu\rangle)$ for $\tau < o^{\mathcal{U}}(\alpha)$.

Now, for each $\delta \in \mathrm{Suc}_{T_2}(\langle\nu\rangle)$, if there are σ,τ such that $j_{\sigma,\tau} = 0$ and $\delta \in X^0_\tau(\langle\nu\rangle)$ then we stipulate that $(T_2)_{\langle\nu,\delta\rangle} := S(\nu,\delta)$ for some tree witnessing it. Otherwise, $(T_2)_{\langle\nu,\delta\rangle} := (T_1)_{\langle\nu,\delta\rangle}$. Note that T_2 is a tree for a condition.

Let $p_2 := \langle t, T_2\rangle$. Clearly, $p_2 \leq^* p_1$.

Suppose that there was a two-step-extension $q \leq p_2$ inside D. Then, $q \leq^* p_2{}^\frown\langle\nu,\delta\rangle$ where $\nu \in Y^\sigma$ and $\delta \in X^{j_{\sigma',\tau}}_\tau(\nu)$ for some $\sigma,\tau,\sigma' < o^{\mathcal{U}}(\alpha)$.

Observe that T^q witnesses that $\delta \in X^0_\tau(\nu)$ ergo $j_{\sigma',\tau} = 0$. Taking any $\nu' \in Y^{\sigma',0}_\tau$ and any $\delta' \in X^0_\tau(\nu')$ we have that

$$p_2{}^\frown\langle\nu',\delta'\rangle = \langle t^\frown\langle\nu',\delta'\rangle, (T_2)_{\langle\nu',\delta'\rangle}\rangle = \langle t^\frown\langle\nu',\delta'\rangle, S(\nu',\delta')\rangle \in D.$$

Let Υ be the p_2-fat tree whose first level is $Y^{\sigma',0}_\tau$ and for each ν' there in $\mathrm{Suc}_\Upsilon(\langle\nu\rangle) = X^0_\tau(\nu')$. It follows that p_2 and Υ witness the desired property.

Once the sequence $\langle p_n \mid n < \omega\rangle$ is defined we let p^* a $\leq^*$-lower bound for the p_n's. Since D is dense there is, for some $n < \omega$, an n-step-extension of p^* inside D. Then, there is an n-step extension of p^n inside D. By the induction hypothesis, there is a p_n-fat tree Υ such that $p_n{}^\frown\vec{\nu} \in D$ for all

$\vec{\nu} \in [\Upsilon]$. Note that $\Upsilon \cap T^{p^*}$ is a p^*-fat tree and that $p^* {}^\frown \vec{\nu} \in D$ for all $\vec{\nu} \in [\Upsilon]$. This completes the proof of the lemma. □

Theorem 3.14. *$\mathbb{Q}^{\alpha^+}_\alpha$ forces "α is regular".*

Proof. Suppose otherwise. Let $p = \langle t, T\rangle \in \mathbb{Q}^{\alpha^+}_\alpha$ and $\dot{f}$ be a $\mathbb{Q}^{\alpha^+}_\alpha$-name for a function such that $p \Vdash_{\mathbb{Q}^{\alpha^+}_\alpha}$ "$\dot{f}\colon \mu \to \alpha$ is cofinal" for some $\mu < \alpha$.

For each $\xi < \mu$ consider the dense open set defined by

$$D_\xi := \{q \in \mathbb{Q}^{\alpha^+}_\alpha \mid q \perp p \vee q \Vdash_{\mathbb{Q}^{\alpha^+}_\alpha} \dot{f}(\xi) < \max(\dot{b}_q)\}.$$

By the Strong Prikry Property (Lemma 3.13) there is $p_\xi \leq^* p$ and Υ_ξ a p_ξ-fat tree of height n_ξ. Since $\mu < \alpha$ and $\langle \mathbb{Q}^{\alpha^+}_\alpha, \leq^*\rangle$ is α-closed, we can arrange $\langle p_\xi \mid \xi < \mu\rangle$ be $\leq^*$-decreasing and p^* a $\leq^*$-lower bound.

For each $\xi < \mu$ and $\vec{\nu} \in \Upsilon_\xi \setminus [\Upsilon_\xi]$ there is $\tau(\xi, \vec{\nu}) < o^{\mathcal{U}}(\alpha) = \alpha^+$ such that

$$\mathrm{Suc}_{\Upsilon_\xi}(\vec{\nu}) \in \mathcal{U}_{\alpha,\tau(\xi,\vec{\nu})}(t^\frown \vec{\nu}).$$

Let $\tau^* := \left(\sup_{\xi<\mu} \sup_{\vec{\nu}\in\Upsilon_\xi} \tau(\xi,\vec{\nu})\right) + 1$ noting that $\tau^* < \alpha^+$.

The following claim is standard, but we include it to convince the reader that our slight variation of coherent sequence is appropriate:

Claim 3.14.1. *For each $\sigma_0 \leq \sigma_1$ and a σ_0-coherent $s \in [\alpha]^{<\omega}$, the map $\pi^s_{\sigma_1,\sigma_0}\colon \alpha \to \alpha$ given by*

$$\pi^s_{\sigma_1,\sigma_0}(\nu) := \min\{x \in b_\nu \mid x > \max(s) \wedge o^{\mathcal{U}}(x) = f_{\sigma_0}(x)\}$$

defines a Rudin-Keisler projection from $\mathcal{U}_{\alpha,\sigma_1}(s)$ to $\mathcal{U}_{\alpha,\sigma_0}(s)$.[11]

Proof. We have to show that whenever $A \in \mathcal{U}_{\alpha,\sigma_0}(s)$, $(\pi^s_{\sigma_1,\sigma_0})^{-1}[A] \in \mathcal{U}_{\alpha,\sigma_1}$. We prove the following stronger statement by induction on σ_1: Whenever $A \in \mathcal{U}_{\alpha,\sigma_0}(s)$ the set of all $\nu < \alpha$ such that either $o^{\mathcal{U}}(\nu) < f_{\sigma_0}(\nu)$ or $\pi_{o^{\mathcal{U}}(\nu),\sigma}(\nu) \in A$ is an element of $\bigcap_{\sigma<\sigma_1} \mathcal{U}_{\alpha,\sigma}(s)$. To this end, we may assume without loss of generality that A only contains points above $\max(s)$.

The statement clearly holds for $\sigma_1 = \sigma_0$ since there is nothing to do. Also, there is nothing to do in limit steps. So suppose that $\sigma_1 = \sigma_1' + 1$. We must show that $A \in \mathcal{U}_{\alpha,\sigma_1'}$. To this end, consider the condition $(s \restriction \sigma_1', T)$, where T is a tree on $[\alpha]^{<\omega}$ with $\mathrm{Suc}_T(\vec{\nu}) \subseteq A$ for each $\vec{\nu} \in T$. By the inductive hypothesis, $(s \restriction \sigma_1', T)$ is a condition in $\mathbb{Q}^{\sigma_1'}_\alpha$. Also, it is easy to see that whenever H is $\mathbb{Q}^\nu_\alpha$-generic containing $(s \restriction \sigma_1', T)$, $j(\pi_{\sigma_1',\sigma_0})(\alpha) \in A$. □

Assume for the time being that for each $\xi < \mu$

$$(\star_\xi) \qquad X_\xi := \{\nu < \alpha \mid \langle \nu^\xi_0, \ldots, \nu^\xi_{n_\xi - 1}\rangle \in [\Upsilon_\xi]\} \in \mathcal{U}_{\alpha,\tau^*}(t)$$

where $\nu^\xi_0 := \pi^t_{\tau^*,\tau(\xi,\varnothing)}(\nu)$ and, recursively, $\nu^\xi_{i+1} := \pi^{t^\frown\langle \nu^\xi_0,\ldots,\nu^\xi_i\rangle}_{\tau^*,\tau(\xi,\langle \nu^\xi_0,\ldots,\nu^\xi_i\rangle)}(\nu)$.

The next claim yields the desired contradiction:

[11] Note that for $\sigma_0 = \sigma_1$, the projection is simply the identity on a measure-one set, since for $o^{\mathcal{U}}(\nu) = f_{\sigma_0}(\nu)$, the only point x in b_ν of order $f_{\sigma_0}(x)$ is ν itself.

Claim 3.14.2. *If $(\star_\xi)$ holds for all $\xi < \mu$ then there is $p^{**} \leq^* p^*$ such that for each $\nu \in \mathrm{Suc}_{T^{p^{**}},\tau^*}(\varnothing)$, $p^{**\frown}\langle\nu\rangle \Vdash_{\mathbb{Q}^{\alpha+}_\alpha} \mathrm{ran}(\dot{f}) < \nu$.*

Proof of claim. Let $p^{**} \leq^* p^*$ be such that $\mathrm{Suc}_{T^{p^{**}},\tau^*}(\varnothing) \subseteq \bigcap_{\xi<\mu} X_\xi$. Let ν be in this set and $\xi < \mu$ arbitrary. Since $\nu \in X_\xi$ it follows that

$$\langle \nu^\xi_0, \dots, \nu^\xi_{n_\xi - 1}\rangle^\frown\langle\nu\rangle \equiv \langle\nu\rangle;$$

namely, both sequences induce the same Prikry/Magidor sequence $b_\nu \cup \{\nu\}$.

As a result, $p^{**\frown}\langle \nu^\xi_0, \dots, \nu^\xi_{n_\xi-1}\rangle^\frown\langle\nu\rangle$ forces the same as $p^{**\frown}\langle\nu\rangle$. However,

$$p^{**\frown}\langle \nu^\xi_0, \dots, \nu^\xi_{n_\xi-1}\rangle^\frown\langle\nu\rangle \leq p_\xi{}^\frown\langle \nu^\xi_0, \dots, \nu^\xi_{n_\xi-1}\rangle \Vdash_{\mathbb{Q}^{\alpha+}_\alpha} \dot{f}(\xi) < \nu^\xi_{n_\xi-1} < \nu.$$

This gives the desired result. □

Claim 3.14.3. *For every $\xi < \mu$, $(\star_\xi)$ holds in $V[G]$.*

Proof of claim. Fix $\xi < \mu$ and assume toward a contradiction that $X_\xi \notin \mathcal{U}_{\alpha,\tau^*}(t)$. Hence, $\alpha \smallsetminus X_\xi \in \mathcal{U}_{\alpha,\tau^*}(t)$ and thus there are $p \in G$ and a $\mathbb{P}_\alpha$-name $\dot{T}$ such that p decides the height of $\dot{\Upsilon}_\xi$ to be n_ξ, $p \Vdash_{\mathbb{P}_\alpha}$ "$\dot{T}$ is a tree on $[\alpha]^{<\omega}$" and

$$p^\frown\langle t, \dot{T}\rangle^\frown j_{\alpha,\tau^*}(p) \smallsetminus \alpha + 1 \Vdash_{j_{\alpha,\tau^*}(\mathbb{P}_\alpha)} \check{\alpha} \notin j_{\alpha,\tau^*}(\dot{X}_\xi).$$

We will show that this is impossible. Work in $V[G]$. By assumption, $\mathrm{Suc}_{\Upsilon_\xi}(\emptyset) \in \mathcal{U}_{\alpha,\tau(\xi,\emptyset)}(t)$. Therefore, there is $\nu_0 \in \mathrm{Suc}_{\Upsilon_\xi}(\emptyset)$ such that $o^{\mathcal{U}}(\nu_0) = f_{\tau(\xi,\emptyset)}(\nu_0)$ and $\nu_0 \in \mathrm{Suc}_T(\emptyset)$ (since $\mathrm{Suc}_T(\emptyset)$ is in every measure $\mathcal{U}_{\alpha,\sigma}(t)$). Inductively, we choose $\langle \nu_0, \dots, \nu_{n_\xi-1}\rangle \in [\Upsilon_\xi]$ such that for every $i < n_\xi$, $o^{\mathcal{U}}(\nu_i) = f_{\tau(\xi,\langle\nu_0,\dots,\nu_{i-1}\rangle)}(\nu_i)$ and $\nu_i \in \mathrm{Suc}_T(\langle\nu_0,\dots,\nu_{i-1}\rangle)$. To see that this is possible, suppose that $\langle\nu_0,\dots,\nu_{i-1}\rangle$ has been defined for $i < n_\xi$. By assumption, $\mathrm{Suc}_{\Upsilon_\xi}(\langle\nu_0,\dots,\nu_{i-1}\rangle) \in \mathcal{U}_{\alpha,\tau(\xi,\langle\nu_0,\dots,\nu_{i-1}\rangle)}(t^\frown\langle\nu_0,\dots,\nu_{i-1}\rangle)$. Furthermore, $\mathrm{Suc}_T(\langle\nu_0,\dots,\nu_{i-1}\rangle) \in \mathcal{U}_{\alpha,\tau(\xi,\langle\nu_0,\dots,\nu_{i-1}\rangle)}(t^\frown\langle\nu_0,\dots,\nu_{i-1}\rangle)$ by the definition of $\mathbb{Q}^{\alpha+}_\alpha$. Therefore there must exist ν_i as required.

By extending p inside G we may assume that p $\mathbb{P}_\alpha$-forces:

- $o^{\mathcal{U}}(\nu_i) = f_{\tau(\xi,\langle\check{\nu}_0,\dots,\check{\nu}_{i-1}\rangle)}(\nu_i)$ and $\nu_i \in \dot{\mathrm{Suc}}_{\Upsilon_\xi}(\langle\nu_0,\dots,\nu_{i-1}\rangle)$.
- $\dot{\mathrm{Suc}}_{\Upsilon_\xi}(\langle\nu_0,\dots,\nu_{i-1}\rangle) \subseteq \check{\alpha}$.

for all $i < n_\xi$.

It follows from the definition of the projection maps that

$$p^\frown\langle t^\frown\langle\nu_0,\dots,\nu_{n_\xi-1}\rangle, \dot{T}_{\langle\nu_0,\dots,x_{n_\xi-1}\rangle}\rangle^\frown(j_{\alpha,\tau^*}(p) \smallsetminus \alpha + 1)$$

forces "$\langle\alpha^\xi_0,\dots,\alpha^\xi_{n_\xi-1}\rangle = \langle\nu_0,\dots,\nu_{n_\xi-1}\rangle$". Indeed, we see inductively that the first element x of $\dot{b}_\alpha$ above $\max(t)$ with $o^{\mathcal{U}}(x) = f_{\tau(\xi,\emptyset)}(x)$ is forced to be ν_0, the first element above $\max(t^\frown\langle\nu_0\rangle)$ with order $\tau(\xi,\langle\nu_0\rangle)$ is forced to be ν_1, and the same holds for the other ν_i's.

As a result the condition above-displayed forces

$$\text{"}\langle\alpha^\xi_0,\dots,\alpha^\xi_{n_\xi-1}\rangle \in [\dot{\Upsilon}_\xi]\text{"}.$$

However we need to show that a suitable extension of this condition forces

$$\text{“}\langle \alpha^\xi_0, \dots, \alpha^\xi_{n_\xi - 1}\rangle \in [j_{\alpha,\tau^*}(\dot{\Upsilon}_\xi)]\text{”}$$

so as to be able to establish that $X_\xi \in \mathcal{U}_{\alpha,\tau^*}(t)$ and get the desired contradiction. To show this last bit we need to prove the following technical claim which first appeared in [HP26a, Claim 7.1.1]:

Subclaim 3.14.3.1. *Suppose that $q \Vdash_{\mathbb{P}_\alpha} \dot{Y} \subseteq \check{\alpha}$. Then,*

$$\{r \leq q \mid j_{\alpha,\tau^*}(r) \Vdash j_{\alpha,\tau^*}(\dot{Y}) \cap \check{\alpha} = \dot{Y}\}$$

is $\mathbb{P}_\alpha$-dense below q.

Proof of subclaim. To streamline the presentation denote j_{α,τ^*} by j and suppose that q is just the trivial condition of $\mathbb{P}_\alpha$.

Let $g \subseteq \mathbb{Q}^{\alpha^+}_\alpha$ a generic filter over $V[G]$. Working inside $V[G*g]$, for each ordinal $\beta < \alpha$ we look at the following set of conditions:

$$D_\beta := \{r \in j(\mathbb{P}_\alpha)/G*g \mid r \Vdash_{j(\mathbb{P}_\alpha)/G*g} \beta \in \check{Y} \leftrightarrow \beta \in j(\dot{Y})_{G*g}\}.$$

Note that D_β is a $\leq^*$-dense open subset of $j(\mathbb{P}_\alpha)/G*g$ for all $\beta < \alpha$.

Let $d_\beta \colon \alpha \to V$ be a function representing D_β; namely, $D_\beta = j(d_\beta)(\alpha)_{G*g}$. Without losing any generality we may assume that $d_\beta(\eta)$ is a $\mathbb{P}_{\eta+1}$-name for a $\leq^*$-dense open subset of the tail poset $\mathbb{P}_\alpha/G_{\eta+1}$.

Our goal is to show that the following set is $\leq^*$-dense:

$$E = \{r \in \mathbb{P}_\alpha \mid \exists C \in \mathrm{Cub}_\alpha \wedge \forall \eta \in C\, (r{\restriction}\eta{+}1 \Vdash_{\mathbb{P}_{\eta+1}} \text{“}r{\restriction}[\eta{+}1,\alpha) \in \textstyle\bigcap_{\delta<\eta} d_\delta(\eta)\text{”})\}.$$

If this is the case then any $r \in E \cap G$ will satisfy that

$$j(r) \Vdash_{j(\mathbb{P}_\alpha)} \dot{Y} = j(\dot{Y}) \cap \alpha.$$

So, fix $r \in \mathbb{P}_\alpha$. We define a sequence of triples $\langle \langle r_\delta, \rho_\delta, C_\delta\rangle \mid \delta < \alpha\rangle$ as follows. Using the *Fusion Lemma* (Lemma 2.28) relative to $\rho_0 := 0$ and the function d_0 we find $r_0 \leq^* p$ such that $r_0 \restriction \nu_0 + 1 = p \restriction \nu_0 + 1$, and a club $C_0 \subseteq \alpha$ (which we may assume $C_0 \cap \mathrm{supp}(r_0) = \emptyset$) for which

$$r_0 \restriction \eta + 1 \Vdash_{\mathbb{P}_{\eta+1}} r_0 \restriction [\eta + 1, \alpha) \in d_0(\eta),$$

for all $\eta \in C_0$.

Suppose that $\langle \langle r_\eta, \rho_\eta, C_\eta\rangle \mid \eta < \delta\rangle$ has been defined.

▶ Case $\delta = \bar{\delta} + 1$: We proceed as before. We invoke the Fusion Lemma with respect to $\rho_\delta := \min(C_{\bar{\delta}} \setminus \rho_{\bar{\delta}} + 1)$ and the function d_δ, thus finding $r_\delta \leq^* r_{\bar{\delta}}$ and a club $C_\delta \subseteq C_{\bar{\delta}}$ such that

- $C_\delta \cap \mathrm{supp}(r_\delta) = \emptyset$,
- $r_\delta \restriction \nu_\delta + 1 = r_{\bar{\delta}} \restriction \rho_\delta + 1$,
- and $r_\delta \restriction \eta + 1 \Vdash_{\mathbb{P}_{\eta+1}} \text{“}r_\delta \restriction [\eta + 1, \alpha) \in d_\delta(\eta)\text{”}$ for all $\eta \in C_\delta$.

► Case δ limit: In this case we stipulate $\rho_\delta = \sup_{\eta<\delta}\rho_\eta$ and take

$$\bar{r}_\delta \restriction \rho_\delta + 1 = \bigcup_{\eta<\delta} r_\eta \restriction \rho_\eta + 1$$

and $\bar{r}_\delta \restriction [\rho_\delta + 1, \alpha)$ a $\leq^*$-lower bound for the sequence of tails $r_\eta \restriction [\rho_\eta + 1, \alpha)$. Let $\bar{r}_\delta$ be $\bar{r}_\delta \restriction (\rho_\delta + 1)^\frown r_\delta \restriction [\nu_{\delta+1}, \alpha)$. Note that $\bar{r}_\delta$ is a legitimate condition in $j(\mathbb{P}_\alpha)$ (i.e., the support of $\bar{p}_\delta$ is nowhere stationary) as witnessed by the club $\bar{C}_\delta := \bigcap_{\bar{\delta}<\delta} C_{\bar{\delta}}$. Once more, appeal to the Fusion Lemma with respect to the ordinal ρ_δ, the function d_δ and the condition $\bar{r}_\delta$, thus finding $r_\delta \leq^* \bar{r}_\delta$ and a club $C_\delta \subseteq \bigcap_{\beta<\delta} C_\beta$ with the properties mentioned above.

We obtain a sequence $\langle \langle r_\delta, \rho_\delta, C_\delta\rangle \mid \delta < \alpha\rangle$. Finally, let

$$r^* := \bigcup_{\delta<\alpha} r_\delta \restriction \rho_\delta + 1,\ C^* := D \cap \triangle_{\delta<\alpha} C_\delta,$$

where D is the club of closure points of the function $R\colon \delta \mapsto \rho_\delta$.

We claim that $r^* \in E$ as witnessed by the club C^*.

Let $\rho \in C$. Then, $\rho = \sup_{\delta<\rho}\rho_\delta$ and $\rho \in \bigcap_{\delta<\rho} C_\delta$. From this we deduce,

$$r^* \restriction \rho + 1 \Vdash_{\mathbb{P}_{\rho+1}} r^* \restriction [\rho+1, \alpha) \leq^* r_\delta \restriction [\rho+1, \alpha) \in d_\delta(\rho)$$

for all $\delta < \rho$. Since $d_\delta(\rho)$ is forced to be open we finally infer

$$r^* \restriction \rho + 1 \Vdash_{\mathbb{P}_{\rho+1}} r^* \restriction [\rho+1, \alpha) \in \bigcap_{\delta<\rho} d_\delta(\rho),$$

for all $\rho \in C$. Thus, we deduce that $r^* \leq^* r$ is in E. □

For each $i < n_\xi$ let $\dot{Y}_i$ denote $\dot{\mathrm{Suc}}_{\Upsilon_\xi}(\langle \nu_0, \dots, \nu_{i-1}\rangle)$. Applying the previous subclaim n_ξ-many times we obtain a decreasing sequence $\langle r_i \mid i < n_\xi\rangle$ of conditions in G such that $r_0 \leq p$ and $j_{\beta,\tau^*}(r_i) \Vdash j_{\alpha,\tau^*}(\dot{Y}_i) \cap \check{\alpha} = \dot{Y}_i$.

Denote $r := r_{n_\xi - 1}$. The above implies that

$$r^\frown\langle t^\frown\langle \nu_0, \dots, \nu_{n_\xi-1}\rangle, \dot{T}_{\langle \nu_0,\dots,x_{n_\xi-1}\rangle}\rangle^\frown(j_{\alpha,\tau^*}(r) \smallsetminus \alpha + 1)$$

forces

$$\text{“}\langle \alpha^\xi_0, \dots, \alpha^\xi_{n_\xi-1}\rangle \in \prod_{i<n_\xi} \left(j_{\alpha,\tau^*}(\dot{\mathrm{Suc}}_{\Upsilon_\xi}(\langle \nu_0, \dots, \nu_{i-1}\rangle)) \cap \alpha\right) \subseteq [j_{\alpha,\tau^*}(\dot{\Upsilon}_\xi)]^{n_\xi}\text{”},$$

as needed. □

This completes the proof of the theorem. □

Proposition 3.15. *Suppose that $\mathbb{Q}$ is an α^+-proper forcing that preserves the regularity of α and does not add bounded subsets to α. Then, $\mathbb{Q}$ is α-distributive. In particular, $\mathbb{Q}^{\alpha^+}_\alpha$ is α-distributive.*

Proof. Let $G \subseteq \mathbb{Q}$ be a generic filter. We show that $({}^\mu\mathrm{Ord})^{V[G]} = ({}^\mu\mathrm{Ord})^V$.

Let $p \in G$ and $\dot{f}$ be a $\mathbb{Q}$-name such that $p \Vdash_{\mathbb{Q}} \dot{f}\colon \mu \to \mathrm{Ord}$. Let $\mathcal{M} \prec H_\Theta$ such that $|\mathcal{M}| = \alpha$, ${}^{<\alpha}\mathcal{M} \subseteq \mathcal{M}$, $\mathbb{Q}, p, \dot{f} \in \mathcal{M}$. For each $\beta < \mu$ consider

$$D_\beta = \{q \in \mathbb{Q} \mid \exists \rho\, (q \Vdash_{\mathbb{Q}} \dot{f}(\beta) = \rho)\}.$$

This is a dense open subset of $\mathbb{Q}$ in $\mathcal{M}$. By properness, there is $q \leq p$, $q \in G$, such that $q \Vdash \forall \beta < \mu\, (D_\beta \cap \mathcal{M} \cap \dot{G} \neq \emptyset)$. By elementarity of $\mathcal{M}$,

$$q \Vdash \mathrm{range}(\dot{f}) \subseteq \mathcal{M}$$

hence range($\dot{f}_G$) $\subseteq \mathcal{M}$. Working in V, let $\varphi\colon \alpha \leftrightarrow \mathcal{M}$ be a bijection. Then, $\varphi^{-1} \circ \dot{f}_G\colon \mu \to \alpha$. Since $\mathbb{Q}$ preserves the regularity of α there is $\xi < \alpha$ such that range(ψ) $\subseteq \xi$. As such, ψ can be coded as a bounded subset of α (for instance, using Gödel's pairing function) thus obtaining $\psi \in V$. Finally, $\dot{f}_G = \varphi \circ \psi$ is in the ground model, as claimed. □

Remark 3.16. By the same argument, if $\mathbb{Q}$ is a forcing which has the α^+-covering property, preserves the regularity of α and does not add bounded subsets to α, then $\mathbb{Q}$ is α-distributive.

3.2. **The iteration when $o^{\mathcal{U}}(\alpha) = \alpha^+$.** We now describe the iterand $\dot{\mathbb{Q}}_\alpha$ for $o^{\mathcal{U}}(\alpha) = \alpha^+$. Our goal is the following: At stage κ, we want to shoot a club through the set of V-regular cardinals below κ. In order to ensure that this preserves the supercompactness of κ, we need to have performed the same forcing already at many cardinals below κ. However, the forcing shooting a club trough the set of V-regular cardinals is typically going to destroy stationary subsets of κ and therefore cannot be κ-closed and will merely be κ-distributive. As merely distributive forcing notions typically cannot be iterated well, we realize our club-shooting forcing as a Prikry-type forcing notion with a κ-closed pure extension ordering. However, while doing so, we need to ensure that this poset remains homogeneous (at least in the sense stipulated in Definition 4.1). To do so, instead of using a single club-shooting forcing, we use a $<\alpha$-supported product (see Remark 3.27 for details). To streamline the presentation we shall adopt the following convention for the scope of this section:

Notation 3.17. We write $\Vdash$ for $\Vdash_{\mathbb{Q}_\alpha^{\alpha^+}}$ and our ground model will be $V[\mathbb{P}_\alpha]$. When we want to refer to the initial ground model V, we shall write V_0.

Definition 3.18. Let $\mathbb{C}_\alpha$ be the following poset: Conditions are partial functions $c\colon \beta^+ \to V$ of size $< \beta$ such that, for every $\xi \in \mathrm{dom}(c)$, $c(\xi)$ is a closed bounded subset of β consisting of V_0-measurable cardinals.

From Fact 2.23, we can see the following:

Lemma 3.19. *Suppose that the collection of V_0-measurable cardinals below α is a fat stationary set. Then $\mathbb{C}_\alpha$ is α-distributive.*

Proof. We can regard $\mathbb{C}_\alpha$ as the product $\mathbb{CS}_\alpha \times \mathbb{C}_\alpha$ and therefore the iteration $\mathbb{CS}_\alpha * \mathbb{C}_\alpha^V$ as well. By Fact 2.23, $\mathbb{CS}_\alpha$ is α-distributive. Let G be $\mathbb{CS}_\alpha$-generic. In $V[G]$, let $C \subseteq \alpha$ be a club consisting of V_0-measurable cardinals. By the α-distributivity of $\mathbb{CS}_\alpha$, $\mathbb{C}_\alpha^V = \mathbb{C}_\alpha^{V[G]}$. Moreover, in $V[G]$, $\mathbb{C}_\alpha$ contains an α-closed dense subset consisting of those $c \in \mathbb{C}_\alpha$ such that for every $\xi \in \mathrm{dom}(c)$, $\max(c(\xi)) \in C$. In particular, $\mathbb{C}_\alpha$ is α-strategically closed in $V[G]$. In particular, $\mathbb{C}_\alpha$ is α-distributive in $V[G]$ and so, since G was arbitrary, $\mathbb{CS}_\alpha * \mathbb{C}_\alpha^V$ is α-distributive in V. This implies the statement of the lemma. □

Now we can define our preliminary forcing $\mathbb{R}_\alpha^0$:

Definition 3.20. Let $\mathbb{R}^0_\alpha := \mathbb{Q}^{\alpha^+}_\alpha * \dot{\mathbb{C}}_\alpha$. For $(p,\dot{c}),(p',\dot{c}') \in \mathbb{R}^0_\alpha$, we let $(p',\dot{c}') \leq_{\mathbb{R}^0_\alpha} (p,\dot{c})$ if $p' \leq_{\mathbb{Q}^{\alpha^+}_\alpha} p$ and $p' \Vdash \dot{c}' \leq_{\dot{\mathbb{C}}_\alpha} \dot{c}$ and we let $(p',\dot{c}') \leq^*_{\mathbb{R}^0_\alpha} (p,\dot{c})$ if $p' \leq^*_{\mathbb{Q}^{\alpha^+}_\alpha} p$ and $p' \Vdash \dot{c}' \leq_{\dot{\mathbb{C}}_\alpha} \dot{c}$.

The next describes the properties of the preliminary triple $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}, \leq^*_{\mathbb{R}^0_\alpha}\rangle$:

Lemma 3.21 (Properties of $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}, \leq^*_{\mathbb{R}^0_\alpha}\rangle$)**.**

(1) $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}, \leq^*_{\mathbb{R}^0_\alpha}\rangle$ *is of Prikry-type.*
(2) $\langle \mathbb{R}^0_\alpha, \leq^*_\alpha\rangle$ *is* α*-strategically-closed.*
(3) There is a projection between $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}\rangle$ *and* $\mathrm{RO}^+(\mathbb{C}_\alpha)$.

Proof. (1). This is obvious.

(2). To establish this it suffices to show that $\langle \mathbb{R}^0_\alpha, \leq^*_{\mathbb{R}^0_\alpha}\rangle$ possesses a $\leq^*_{\mathbb{R}^0_\alpha}$-dense subset that is α-directed-closed. First, we identify the $\leq^*_{\mathbb{R}^0_\alpha}$-dense set:

Claim 3.21.1. *The set*

$$\{(p,\dot{c}) \in \mathbb{R}^0_\alpha \mid p \Vdash \forall i \in \mathrm{dom}(\dot{c})\,(\max(\dot{c}(i)) \in \dot{b}_\alpha)\}$$

is $\leq^*_{\mathbb{R}^0_\alpha}$*-dense.*

Proof of claim. Let $(p,\dot{c}) \in \mathbb{R}^0_\alpha$. Since p forces "α is regular" (Theorem 3.14), "$\dot{b}_\alpha$ is a club on α" and "$|\mathrm{dom}(\dot{c})| < \alpha$", the maximality principle gives a $\mathbb{Q}^{\alpha^+}_\alpha$-name $\dot{\gamma}$ that p forces to be above every $\dot{c}(i)$ and inside $\dot{b}_\alpha$. Let $\dot{d}$ be the standard name for the condition in $\dot{\mathbb{C}}_\alpha$ with same domain as $\dot{c}$ such that it is forced that $\dot{d}(i) = \dot{c}(i) \cup \{\dot{\gamma}\}$. It follows that $(p,\dot{d})$ is a condition within the set under consideration such that $(p,\dot{d}) \leq^*_{\mathbb{R}^0_\alpha} (p,\dot{c})$. □

For the scope of this prove denote by D the above-displayed set. We claim that $\langle D, \leq^*_{\mathbb{R}^0_\alpha}\rangle$ is α-directed-closed: Let $\{(\langle t_i, T_i\rangle, \tau_i) \mid i < j\}$ be a directed subset of D of size $<\alpha$. Then, the stems t_i are equivalent. This implies that the trees T_i, T_k split according to the same measures (see Remark 3.9) and therefore $T_i \cap T_k$ is a valid tree. As a result, $p_j := \langle t_0, \bigcap_{i<j} T_i\rangle$ provides a $\leq^*$-lower bound for all the $\langle t_i, T_i\rangle$'s. It follows that

$$p_j \Vdash \forall \xi \in \bigcup_{i<j} \mathrm{supp}(\tau_i) \left(\sup_{i<j} \max(\tau_i(\xi)) \in \dot{b}_\alpha \right)$$

Since $\dot{b}_\alpha$ is forced to contain only V-measurable cardinals, p_j forces that for all $\xi \in \bigcup_{i<j} \mathrm{supp}(\tau_i)$, the above supremum can be appended to $\bigcup_{i<j} \tau_i(\xi)$. Letting τ be a name such that

$$p_j \Vdash \forall \xi \in \bigcup_{i<j} \tau(\xi) = \bigcup_{i<j} \tau_i(j) \cup \sup_{i<j} \max(\tau_i(\xi))$$

we deduce that (p_j,τ) is a $\leq^*_{\mathbb{R}^0_\alpha}$-lower bound in D.

(3). We show that every $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}\rangle$-generic G induces a $\mathrm{RO}^+(\mathbb{C}_\alpha)$-generic. Once this is established the projection is obtained as follows: Let $\dot{G}^*$ be the

standard $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}\rangle$-name for the induced $\mathbb{C}_\alpha$-generic. Standard forcing facts (see e.g., [DPT25, §2]) show that the map

$$a \mapsto \bigwedge\{c \in \mathrm{RO}^+(\mathbb{C}_\alpha) \mid a \Vdash_{\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}\rangle} \check{c} \in \dot{G}^*\}$$

is a projection.

Let $G * H$ be a $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}\rangle$-generic filter and work in $V[G*H]$. Define

$$G^* := \{c \in \mathbb{C}_\alpha \mid \exists a \in G\,(a = (p, \check{c}))\}.$$

Easily G^* is a filter. To show genericity, we let$D \subseteq \mathbb{C}_\alpha$ be a dense set and we verify that $\{(p,\check{c}) \in \mathbb{R}^0_\alpha \mid c \in D\}$ is dense in $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}\rangle$. Here the α-distributivity of $\mathbb{Q}^{\alpha^+}_\alpha$ (see Proposition 3.15) plays a key role – indeed, it ensures that $\mathbb{C}^{V[G]}_\alpha = \mathbb{C}^V_\alpha$. As a result, the trivial condition of $\mathbb{Q}^{\alpha^+}_\alpha$ forces "$\check{D}$ is dense in $\dot{\mathbb{C}}_\alpha$". Given $(p,\dot{c}) \in \mathbb{R}^0_\alpha$ we can use the previous observation to find an extension of it $(q,\check{d})$ such that $q \Vdash \check{d} \in D$, as sought. □

Definition 3.22. Let $\pi\colon \mathbb{R}^0_\alpha \to \mathrm{RO}^+(\mathbb{C}_\alpha)$ denote the map

$$(p,\dot{c}) \mapsto \bigwedge\{b \in \mathrm{RO}^+(\mathbb{C}_\alpha) \mid (p,\dot{c}) \Vdash_{\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}\rangle} \check{b} \in \dot{G}^*\}$$

The following lemma will be helpful later, since it shows in essence that the definition of π does not depend too much on the first coordinate.

Lemma 3.23. *For every $(p,\dot{c}) \in \mathbb{R}^0_\alpha$,*

$$\pi((p,\dot{c})) = \bigwedge\{b \in \mathrm{RO}^+(\mathbb{C}_\alpha) \mid p \Vdash_{\mathbb{Q}^{\alpha^+}_\alpha} \dot{c} \leq_{\mathrm{RO}^+(\dot{\mathbb{C}}_\alpha)} \check{b}\}$$

Proof. Recall that $\dot{G}^*$ was a name for the upward closure of $\pi_2[\Gamma_{\mathbb{R}^0_\alpha}]$. It suffices to show that for every $(p,\dot{c}) \in \mathbb{R}^0_\alpha$, $(p,\dot{c}) \Vdash \check{b} \in \dot{G}^*$ if and only if $p \Vdash_{\mathbb{Q}^{\alpha^+}_\alpha} \dot{c} \leq_{\mathrm{RO}^+(\mathbb{C}_\alpha)} \check{b}$. The backward direction is immediate. So suppose that $(p,\dot{c}) \Vdash_{\mathbb{R}^0_\alpha} \check{b} \in \dot{G}^*$ and suppose toward a contradiction that $p \nVdash_{\mathbb{Q}^{\alpha^+}_\alpha} \dot{c} \leq \check{b}$. In particular, there is $p' \leq_{\mathbb{Q}^{\alpha^+}_\alpha} p$ such that $p' \Vdash_{\mathbb{Q}^{\alpha^+}_\alpha} \dot{c} \not\leq_{\mathrm{RO}^+(\dot{\mathbb{C}}_\alpha)} \check{b}$. Letting $\dot{c}'$ be a name for $\dot{c} - \check{b}$, which is forced by p' to be nonzero, we have

$$p' \Vdash_{\mathbb{Q}^{\alpha^+}_\alpha} \dot{c}' \leq_{\mathrm{RO}^+(\dot{\mathbb{C}}_\alpha)} \dot{c} \wedge \dot{c}' \perp_{\mathrm{RO}^+(\dot{\mathbb{C}}_\alpha)} \check{b}$$

But then $(p',\dot{c}') \leq_{\mathbb{R}^0_\alpha} (p,\dot{c})$ forces $\check{b} \notin \dot{G}^*$, a contradiction. □

Now we are ready to define our actual forcing $\mathbb{R}_\alpha$.

Definition 3.24. Define the forcing $\mathbb{R}_\alpha$ as follows: $\mathbb{R}_\alpha := \mathbb{R}^0_\alpha$. For $r, r' \in \mathbb{R}_\alpha$, we let $r' \leq_{\mathbb{R}_\alpha} r$ if $\pi(r') \leq_{\mathrm{RO}^+(\mathbb{C}_\alpha)} \pi(r)$ and we let $r' \leq^*_{\mathbb{R}_\alpha} r'$ if $r' \leq^*_{\mathbb{R}^0_\alpha} r$.

Clearly $\leq_{\mathbb{R}^0_\alpha} \subseteq \leq_{\mathbb{R}_\alpha}$ because π is a projection. In particular, $\leq^*_\alpha \subseteq \leq_\alpha$, so it makes sense to wonder if $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}, \leq^*_{\mathbb{R}_\alpha}\rangle$ is of Prikry-type.

Lemma 3.25 (Prikry property)**.** *Given $a \in \mathbb{R}_\alpha$ and φ a sentence in the language of forcing of $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}\rangle$, there is $b \leq^*_{\mathbb{R}_\alpha} a$ such that $b \parallel_{\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}\rangle} \varphi$.*

Proof. Let $a \in \mathbb{R}_\alpha$ and φ be as before. For each $i < 2$, let

$$D_i = \{b \in \mathbb{R}_\alpha \mid \pi(b) \Vdash^i_{\mathrm{RO}^+(\mathbb{CS}_\alpha)} \varphi\}.$$

Clearly, $D_0 \cup D_1$ is dense. The Prikry property of $\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha}, \leq^*_{\mathbb{R}^0_\alpha} \rangle$ yields a condition $b \leq^*_{\mathbb{R}^0_\alpha} a$ with

$$b \Vdash^i_{\langle \mathbb{R}^0_\alpha, \leq_{\mathbb{R}^0_\alpha} \rangle} \dot{G} \cap \check{D}_0 \neq \emptyset.$$

Since $D_0 \cup D_1$ is dense, if b forces the negation of the above sentence then b must force "$\dot{G} \cap \check{D}_1 \neq \emptyset$". Suppose, without loss of generality, that $b \Vdash_{\mathbb{R}^0_\alpha} \dot{G} \cap \check{D}_0 \neq \emptyset$ (the other case is treated analogously).

We claim that

$$\pi(b) \Vdash_{\mathrm{RO}^+(\mathbb{C}_\alpha)} \varphi,$$

which by the definition of $\leq_{\mathbb{R}_\alpha}$ would yield $b \Vdash_{\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha} \rangle} \varphi$.

Otherwise, let $r \leq_{\mathrm{RO}^+(\mathbb{CS}_\alpha)} \pi(b)$ forcing $\neg\varphi$. Let $b' \leq_{\mathbb{R}^0_\alpha} b$ be such that $\pi(b') \leq_{\mathrm{RO}^+(\mathbb{CS}_\alpha)} r$. In particular, $\pi(b')$ forces $\neg\varphi$. Since $b' \leq_{\mathbb{R}^0_\alpha} b$ and the latter forces "$\dot{G} \cap \check{D}_0 \neq \emptyset$" in $\mathbb{R}^0_\alpha$, there must be $b'' \in D_0$, compatible with b, such that $\pi(b'')$ forces φ. But this is a contradiction, because $\pi(b)$ and $\pi(b'')$ are compatible, yet they force contradictory information. □

Lemma 3.26 (Properties of $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}, \leq^*_{\mathbb{R}_\alpha} \rangle$)**.**

(1) $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}, \leq^*_{\mathbb{R}_\alpha} \rangle$ *is of Prikry-type and* $\langle \mathbb{R}_\alpha, \leq^*_{\mathbb{R}_\alpha} \rangle$ *is* α*-strategically-closed.*

(2) $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}, \leq^*_{\mathbb{R}_\alpha} \rangle$ *is forcing equivalent to* $\mathbb{C}_\alpha$*.*

(3) $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha}, \leq^*_{\mathbb{R}_\alpha} \rangle$ *is* α^+*-cc and* α*-distributive.*

In particular, forcing with $\langle \mathbb{R}^0_\alpha, \leq_\alpha \rangle$ *preserves cardinals and adds a club on* α *consisting of* V_0*-regulars.*

Proof. (1). The Prikry property is precisely Lemma 3.25. The closure of of $\langle \mathbb{R}_\alpha, \leq^*_{\mathbb{R}_\alpha} \rangle$ follows from Lemma 3.21, using the fact that that poset is literally equal to $\langle \mathbb{R}^0_\alpha, \leq^*_{\mathbb{R}^0_\alpha} \rangle$.

(2). Since $\mathbb{C}_\alpha$ is forcing equivalent to $\mathrm{RO}^+(\mathbb{C}_\alpha)$ it suffices to check that $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha} \rangle$ is forcing equivalent to the latter. To justify this we show that the map $d\colon \langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha} \rangle \to \mathrm{RO}^+(\mathbb{C}_\alpha)$ given by $a \mapsto \pi(a)$ is a dense embedding. Clearly, d is order-preserving and its range is dense in $\mathrm{RO}^+(\mathbb{C}_\alpha)$ because π was a projection. The map also preserves incompatibilities: Suppose that $\pi(a)$ is $\leq_{\mathrm{RO}^+(\mathbb{C}_\alpha)}$-compatible with $\pi(b)$ and $c \in \mathrm{RO}^+(\mathbb{C}_\alpha)$ witnessing this. Since π was a projection, there is a condition $u \leq_{\mathbb{R}^0_\alpha} a$ such that $\pi(u) \leq_{\mathrm{RO}^+(\mathbb{C}_\alpha)} c$. Then, by definition, $u \leq_\alpha a, b$.

(3). Since $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha} \rangle$ is forcing equivalent to $\mathbb{C}_\alpha$, if the latter is α^+-cc and α-distributive then the same would be true for the former. Certainly, $\mathbb{C}_\alpha$ is α^+-cc. The α-distributivity of $\mathbb{C}_\alpha$ follows from Lemma 3.19 with the proviso that $\mathrm{Reg}^{V_0} \cap \alpha$ is a fat stationary set (see Definition 2.22): Let $D \subseteq \alpha$ be a club and $\beta < \alpha$. Since $o^{\mathcal{U}}(\alpha) = \alpha^+$, $S = \{\delta < \alpha \mid o^{\mathcal{U}}(\delta) \geq \beta\}$ is stationary

in V_0. Let $D_0 \subseteq D$ be a club in V_0 (see Lemma 2.31). Then, $\mathrm{acc}(D_0) \cap S$ is non-empty and every δ therein carries a Prikry/Magidor sequence (relative to V_0) $b_\delta \subseteq \delta$ with order-type $\geq \beta$. Therefore, $b_\delta \subseteq^* D \cap (\mathrm{Reg}^{V_0} \cap \alpha)$ is closed with order-type $\geq\beta$. □

It is easy to show that $\langle \mathbb{R}_\alpha, \leq_{\mathbb{R}_\alpha} \rangle$ contains a $\leq_{\mathbb{R}_\alpha}$-dense weakly homogeneous subset, i.e., $\{(p,\dot{c}) \in \mathbb{R}_\alpha \mid \exists c \in \mathbb{C}_\alpha\, (p \Vdash \check{c} = \dot{c})\}$. However the existence of this set is not enough to carry out the proof of homogeneity of the iteration $\mathbb{P}_{\kappa+1}$. Indeed, the issue is that in extensions in nonstationary support iterations must use the $\leq^*$-order in all but finitely many coordinates in the support of the condition that is being extended. If at various stages $\alpha \leq \kappa$ with $o^{\mathcal{U}}(\alpha) = \alpha^+$ we allow ourselves to take $\leq$-extensions after ω-many steps there is a worry that the constructed sequence is not longer a condition in $\mathbb{P}_{\kappa+1}$. This issue is resolved here by taking the $<\alpha$-support product of α^+-many copies of $\mathbb{CS}_\alpha$. Intuitively, this choice confer us "enough room" to shuffle around conditions in the $\mathbb{C}_\alpha$-part of a condition.

Remark 3.27 (**Why we use many copies of club shooting**). Let us illustrate why the more natural approach, using a single copy of $\mathbb{CS}_\alpha$ in $\mathbb{R}^0_\alpha$, does not seem to work. Work in $V[G_\alpha]$, where $G_\alpha \subseteq \mathbb{P}_\alpha$ is V-generic, and let $\dot{\eta}_i$ be the standard $\mathbb{Q}^{\alpha^+}_\alpha$-name for the ith member of $\dot{b}_\alpha$. Consider the conditions $a = (\mathbb{1}, \{\dot{\eta}_0\})$ and $b = (\mathbb{1}, \{\dot{\eta}_1\})$ of $\mathbb{R}^0_\alpha$. They have no common direct extension: such a condition $(q,\dot{c})$ would force $\dot{c}$ to end-extend both $\{\dot{\eta}_0\}$ and $\{\dot{\eta}_1\}$, so $\dot{\eta}_0 = \min(\dot{c}) = \dot{\eta}_1$. Thus the Ben-Neria–Unger argument needs direct extensions $a^* \leq^*_\alpha a$ and $b^* \leq^*_\alpha b$, and an automorphism σ of $\mathbb{R}^0_\alpha$ for both $\leq_\alpha$ and $\leq^*_\alpha$, such that $\sigma(a^*)$ and b^* have a common direct extension.

The first option is $\sigma(q,\dot{c}) = (q, \Gamma(\dot{c}))$ for an automorphism $\Gamma \in V[G_\alpha]$ of $\mathbb{CS}_\alpha$. Write $a^* = (q,\dot{c}_0)$, and let $(r,\dot{e})$ be a common direct extension of $\sigma(a^*) = (q,\Gamma(\dot{c}_0))$ and b^*. Then r is a direct extension of $\mathbb{1}$. As $\dot{c}_0$ end-extends $\{\dot{\eta}_0\}$, and Γ preserves end-extension and fixes $\emptyset$, the set $\Gamma(\{\dot{\eta}_0\})$ is a nonempty initial segment of $\Gamma(\dot{c}_0)$, and hence of $\dot{e}$. Since $\dot{e}$ also end-extends $\{\dot{\eta}_1\}$, we get $r \Vdash \min(\Gamma(\{\dot{\eta}_0\})) = \dot{\eta}_1$. That is, r forces $\dot{\eta}_1 = F(\dot{\eta}_0)$, where $F(\nu) := \min(\Gamma(\{\nu\}))$ is a function in $V[G_\alpha]$. No direct extension of $\mathbb{1}$ forces this, because once the first point of b_α is fixed, the second still ranges over a measure-one set.[12]

The other option is to let the automorphism of $\mathbb{CS}_\alpha$ depend on the auxiliary generic of $\mathbb{Q}^{\alpha^+}_\alpha$. For instance, let $\dot{\Gamma}$ name the automorphism of $\mathbb{CS}_\alpha$ that swaps the cones above $\{\dot{\eta}_0\}$ and $\{\dot{\eta}_1\}$, and put $\sigma(q,\dot{c}) = (q, \dot{\Gamma}(\dot{c}))$. Then σ preserves $\mathbb{R}_\alpha$ and $\leq^*_{\mathbb{R}_\alpha}$, and $\sigma(a) = b$. However, σ does not preserve $\leq_{\mathbb{R}_\alpha}$, which only sees the projection π to $\mathrm{RO}^+(\mathbb{CS}_\alpha)$. To see this, fix $\mu < \nu$ with $o^{\mathcal{U}}(\mu) = o^{\mathcal{U}}(\nu) = 0$, and let $q_0, q_1 \in \mathbb{Q}^{\alpha^+}_\alpha$ have stems $\langle \nu \rangle$ and $\langle \mu, \nu \rangle$. The conditions $x = (q_0, \{\check{\nu}\})$ and $y = (q_1, \{\check{\nu}\})$ belong to $\mathbb{R}_\alpha$ and $\pi(x) = \pi(y) = \{\nu\}$,

[12] Indeed, write $r = \langle \emptyset, T \rangle$, pick $\nu \in \mathrm{Suc}_T(\emptyset)$ with $o^{\mathcal{U}}(\nu) = 0$, and pick distinct $\nu', \nu'' \in \mathrm{Suc}_T(\langle \nu \rangle)$ with $o^{\mathcal{U}}(\nu') = o^{\mathcal{U}}(\nu'') = 0$. Then $r \frown \langle \nu, \nu' \rangle$ and $r \frown \langle \nu, \nu'' \rangle$ both force $\dot{\eta}_0 = \nu$, but they force $\dot{\eta}_1 = \nu'$ and $\dot{\eta}_1 = \nu''$ respectively.

so they are $\leq_{\mathbb{R}_\alpha}$-equivalent. But q_0 forces $\{\nu\} = \{\dot\eta_0\}$, so $\sigma(x) = (q_0, \{\dot\eta_1\})$, and every possible value of its second coordinate has minimum greater than ν. On the other hand, q_1 forces $\{\nu\} = \{\dot\eta_1\}$, so $\sigma(y) = (q_1, \{\check\mu\})$ and $\pi(\sigma(y)) = \{\mu\}$. Hence $\sigma(x)$ and $\sigma(y)$ are $\leq_{\mathbb{R}_\alpha}$-incompatible. In general, an automorphism σ for $\leq_\alpha$ must satisfy that $\pi(\sigma(x))$ depends only on $\pi(x)$, and a $\dot\Gamma$ depending on the auxiliary generic breaks this.

Using α^+ coordinates avoids both problems: a ground-model permutation moves $\{\dot\eta_0\}$ to a different coordinate, so no automorphism has to send $\dot\eta_0$ to $\dot\eta_1$. This idea lies at the core of Claim 4.16.6.

3.3. **Relation with the constructions of Gitik and Ben-Neria–Unger.** It may be useful to explain more precisely how the forcing developed here differs from the constructions on which it is based. Our forcing combines two ingredients: Gitik's method [Git99] for realizing club-shooting through the ground-model regular cardinals as a Prikry-type forcing, and the nonstationary support iteration of Ben-Neria–Unger [BNU17], whose homogeneity properties make it possible to obtain a precise computation of HOD. Neither construction, however, can be used verbatim for our purposes.

There are two difficulties on the Gitik side. In the limiting case $o(\alpha) = \alpha$, the forcing of [Git99] is assembled from α many cells of the form

$$\mathbb{Q}^\beta_\alpha * \mathbb{CS}(\mathrm{Reg}^V \cap \alpha),$$

and the direct-extension order in the βth cell has only the corresponding β-closure. Thus one does not obtain uniformly α-closed direct extensions at stage α. Gitik is able to compensate for this loss of closure by exploiting the Easton support of his iteration; this feature enters essentially, for instance, in the inductive construction of [Git99, Claim 1.15.1], where he shows that cardinals are preserved. The same argument is not available for a nonstationary-support iteration, since the support of a condition may be unbounded in regular cardinals and the fusion arguments require simultaneous control of direct extensions along such supports. We therefore redesign the local forcings so that they admit substantially stronger closure in the direct-extension order, and extend the construction to coherent sequences with Mitchell order as high as $o(\alpha) = \alpha^+$(see §3.1).[13]

A different difficulty arises on the Ben-Neria–Unger side. Their nonstationary support machinery provides exactly the global fusion and homogeneity framework that we need, but the local forcings considered in [BNU17] live in a regime of Mitchell orders strictly below the relevant cardinal. This is not sufficient for our application, where the top large cardinal must survive and where club shooting at the top has to be supported by many corresponding high-order stages below it. We therefore have to incorporate Gitik-type club shooting at stages of large Mitchell order while preserving

[13]The need to allow high Mitchell orders is not a whim: Mitchell [Mit01] shows that producing a club of former regulars while retaining measurability of κ requires consistency strength $o(\kappa) = \kappa$.

the Ben-Neria–Unger homogeneity analysis. This is particularly delicate because the most natural local forcing–using a single copy of $\mathbb{CS}(\mathrm{Reg}^V \cap \alpha)$–does not provide enough flexibility for the required automorphism argument (see Remark 3.27).

Thus the forcing construction developed here is not simply the juxtaposition of the two earlier constructions. We simultaneously strengthen the local Prikry-type forcings and strengthen the homogeneity mechanism so that these more complicated local forcings can still be iterated with nonstationary support. It is this combination that allows us to shoot a club of ground-model measurable cardinals while preserving the top large cardinal–in our application, HOD-supercompactness–and at the same time retain enough homogeneity to identify the HOD of the final model.

4. Homogeneity

In this section we establish that the iteration $\mathbb{P}_{\kappa+1}$ is almost homogeneous (Theorem 4.10). Our automorphism will be built up by an inductive construction. Commonly, this is done by using names for automorphisms of $\dot{\mathbb{Q}}_\beta$ and requiring that the previously built automorphism of $\mathbb{P}_\beta$ preserves $\dot{\mathbb{Q}}_\beta$ (see [DF08, BNU17, BNH23]). However, in our construction, it is vital that $\dot{\mathbb{Q}}_\beta$ is moved by the previously constructed automorphism of $\mathbb{P}_\beta$. Therefore, we will instead consider *isomorphisms* between $\dot{\mathbb{Q}}_\beta$ and the preimage of $\dot{\mathbb{Q}}_\beta$ under the previously constructed automorphism of $\mathbb{P}_\beta$. Furthermore, since we are using a Magidor-style iteration, where the definition of even the regular order on the iteration appeals to the pure extension ordering on the iterands, it is instrumental that these isomorphisms are forced to preserve both $\leq_{\dot{\mathbb{Q}}_\alpha}$ *and* $\leq^*_{\dot{\mathbb{Q}}_\alpha}$. All of these ideas are captured in the following definition, implicit in Ben-Neria–Unger's work [BNU17]:

Definition 4.1. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha \leq \varrho \rangle$ be a nonstationary support iteration of Prikry-type forcings. Let $(\dot{\pi}_\beta, \dot{D}_\beta)_{\beta<\rho}$ be a sequence. We define the objects $\mathcal{D}^\alpha(\vec{D})$ and $\Gamma^\alpha_{\vec{\pi}}$ by induction on $\alpha \leq \rho$, letting

$$\mathcal{D}(\vec{D}) := \mathcal{D}^\varrho(\vec{D}) \text{ and } \Gamma_{\vec{\pi}} := \Gamma^\varrho_{\vec{\pi}}.$$[14]

Suppose $\mathcal{D}^\beta(\vec{D})$ and $\Gamma^\beta_{\vec{\pi}}$ have been defined for all $\beta < \alpha$.

Case $\alpha = 0$: Let $\mathcal{D}^0(\vec{D}) := \mathbb{P}_0 = \{\emptyset\}$ and $\Gamma^0_{\vec{\pi}} \colon \{\emptyset\} \to \{\emptyset\}$.

Case $\alpha = \beta + 1$: Suppose the following:

- $\dot{D}_\beta$ is forced to be a $\leq^*_{\dot{\mathbb{Q}}_\beta}$-dense subset of $\dot{\mathbb{Q}}_\beta$
- $\dot{\pi}_\beta$ is forced to be an isomorphism between $(\dot{D}_\beta, \leq_{\dot{\mathbb{Q}}_\beta}, \leq^*_{\dot{\mathbb{Q}}_\beta})$ and $((\Gamma^\beta_{\vec{\pi}})^{-1})^*((\dot{D}_\beta, \leq_{\dot{\mathbb{Q}}_\beta}, \leq^*_{\dot{\mathbb{Q}}_\beta}))$ such that $\Vdash \dot{\pi}_\beta \circ (\Gamma^\beta_{\vec{\pi}})^*(\dot{\pi}_\beta) = \mathrm{id}_{(\Gamma^\beta_{\vec{\pi}})^*(\dot{D}_\beta)}$.

[14] We will later show inductively that $\mathcal{D}^\alpha(\vec{D})$ is an open dense subset of $\mathbb{P}_\alpha$ and $\Gamma^\alpha_{\vec{\pi}}$ is an automorphism of $\mathcal{D}^\alpha(\vec{D})$ so that the following definition makes sense.

If this is not the case, we stop the construction.

Suppose the two statements above hold. We will suppose that it is forced that $\mathbb{1}_{\dot{\mathbb{Q}}_\beta} \in \dot{D}_\beta$. If this is not the case, we simply add it and let $\dot{\pi}_\beta$ map $\mathbb{1}_{\dot{\mathbb{Q}}_\beta}$ to itself. Let $\mathcal{D}^\alpha(\vec{D})$ consist of all $p \in \mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$ such that $p \restriction \beta \in \mathcal{D}^\beta(\vec{D})$ and $p \Vdash p(\beta) \in \dot{D}_\beta$. Define $\Gamma^\alpha_{\vec{\pi}}$ as follows: For every $p \in \mathcal{D}^\alpha(\vec{D})$,

$$\Gamma^\alpha_{\vec{\pi}}(p) = (\Gamma^\beta_{\vec{\pi}}(p \restriction \beta), (\Gamma^\beta_{\vec{\pi}})^*(\dot{\pi}_\beta(p(\beta)))).$$[15]

Case α is a limit: Let $\mathcal{D}^\alpha(\vec{D})$ consist of all conditions $p \in \mathbb{P}_\alpha$ such that $p \restriction \beta \in \mathcal{D}^\beta(\vec{D})$ for all $\beta < \alpha$. Define $\Gamma^\alpha_{\vec{\pi}}$ by letting, for $p \in \mathcal{D}^\alpha(\vec{D})$, $\Gamma^\alpha_{\vec{\pi}}(p)$ be such that for every $\beta < \alpha$,

$$\Gamma^\alpha_{\vec{\pi}}(p)(\beta) = \Gamma^{\beta+1}_{\vec{\pi}}(p \restriction \beta + 1)(\beta).$$

Finally, we say that $(\dot{\pi}_\beta, \dot{D}_\beta)_{\beta<\rho}$ is an *automorphism sequence* if the construction can proceed until ρ.

Remark 4.2. It is easy to see that the following holds:

(1) $\mathcal{D}(\vec{D})$ consists of all $p \in \mathbb{P}_\rho$ such that for every $\beta < \rho$,
$$p \restriction \beta \Vdash p(\beta) \in \dot{D}_\beta.$$
(2) The automorphisms $\Gamma^\beta_{\vec{\pi}}$ for $\beta \leq \rho$ are coherent in the following sense: Whenever $\beta < \alpha \leq \rho$ and $p \in \mathcal{D}^\alpha(\vec{D})$,
$$\Gamma^\beta_{\vec{\pi}}(p \restriction \beta) = \Gamma^\alpha_{\vec{\pi}}(p) \restriction \beta.$$

It turns out that every automorphism sequence $(\vec{\pi}, \vec{D})$ gives rise to a $\leq^*$-dense subset $\mathcal{D}(\vec{D})$ of $\mathbb{P}_\alpha$ and a self-inverse automorphism of $\mathcal{D}(\vec{D})$ modulo equivalence.

Lemma 4.3. *Let $(\vec{\pi}, \vec{D}) = (\dot{\pi}_\beta, \dot{D}_\beta)_{\beta<\rho}$ be an automorphism sequence. Then the following hold:*

(1) $\mathcal{D}(\vec{D})$ is a $\leq^*$-dense subset of $\mathbb{P}_\rho$;
(2) $\Gamma_{\vec{\pi}}$ is a self-inverse automorphism of $\mathcal{D}(\vec{D})$ modulo equivalence with respect to both $\leq$ and $\leq^*$.

Proof. We prove the statements one-by-one.

For (1), let $p \in \mathbb{P}_\rho$. Using the maximum principle, let $q \in \mathbb{P}_\rho$ be a condition with $\mathrm{supp}(q) = \mathrm{supp}(p)$ such that, for every $\beta \in \mathrm{supp}(p)$,

$$\Vdash_{\mathbb{P}_\beta} q(\beta) \in \dot{D}_\beta \wedge q(\beta) \leq^*_{\dot{\mathbb{Q}}_\beta} p(\beta).$$

Clearly, $q \leq^*_{\mathbb{P}_\rho} p$ and $q \in \mathcal{D}(\vec{D})$.

[15] In [DF08, Lemma 4], the authors define the lifted automorphism the other way around, i.e. as $\dot{\pi}_\beta((\Gamma^\beta_{\vec{\pi}})^*(\dot{q}))$. However, their case is different, since $(\Gamma^\beta_{\vec{\pi}})^*$ does not move $\dot{\mathbb{Q}}_\beta$, while in our case it is important that $(\Gamma^\beta_{\vec{\pi}})^*$ *does* move $\dot{\mathbb{Q}})_\beta$.

To show (2), we prove by induction on $\alpha \leq \rho$ that $\Gamma^\alpha_{\vec{\pi}}$ is a self-inverse automorphism of $\mathcal{D}^\alpha(\vec{D})$ modulo equivalence. Suppose the statement holds for all $\beta < \alpha$.

Case $\alpha = 0$: This is easy.

Case $\alpha = \beta + 1$: For notational simplicity, we write $\Gamma := \Gamma^{\beta'}_{\vec{\pi}}$.

(1) $\Gamma^\alpha_{\vec{\pi}}$ **preserves the orders:** Suppose that $p', p \in \mathcal{D}(\vec{D})$ and $p' \leq_{\mathbb{P}_\alpha} p$, i.e. $p' \restriction \beta \leq_{\mathbb{P}_\beta} p \restriction \beta$ and $p' \restriction \beta \Vdash_{\mathbb{P}_{\beta'}} p'(\beta) \leq_{\dot{D}_\beta} p(\beta)$. It follows that $\Gamma(p' \restriction \beta) \leq_{\mathbb{P}_\beta} \Gamma(p \restriction \beta)$ by the inductive hypothesis. By the hypothesis on $\dot{\pi}_\beta$, we have $p' \restriction \beta \Vdash \dot{\pi}_\beta(p'(\beta) \leq_{(\Gamma^{-1})^*(\dot{D}_\beta)} \dot{\pi}_\beta(p(\beta))$. But then we have $\Gamma(p' \restriction \beta) \Vdash \Gamma^*(\dot{\pi}_\beta(p'(\beta))) \leq_{\dot{D}_\beta} \Gamma^*(\dot{\pi}_\beta(p(\beta)))$ as desired.

The same argument applies when replacing $\leq_{\dot{\mathbb{Q}}_\beta}$ with $\leq^*_{\dot{\mathbb{Q}}_\beta}$.

(2) $\Gamma^\alpha_{\vec{\pi}}$ **maps from $\mathcal{D}^\alpha(\vec{D})$ into $\mathcal{D}^\alpha(\vec{D})$:** Suppose that $p \in \mathcal{D}^\alpha(\vec{D})$. Thus, $p \restriction \beta \in \mathcal{D}^\beta(\vec{D})$, implying $\Gamma(p \restriction \beta) \in \mathcal{D}^\beta(\vec{D})$ by the inductive assumption. Furthermore, $p \restriction \beta \Vdash p(\beta) \in \dot{D}_\beta$. By the assumption, $p \Vdash \dot{\pi}_\beta(p(\beta)) \in (\Gamma^{-1})^*(\dot{D}_\beta)$. But then $\Gamma(p \restriction \beta) \Vdash \Gamma^*(\dot{\pi}_\beta(\dot{q})) \in \dot{D}_\beta$. This easily implies that $\Gamma^\alpha_{\vec{\pi}}(p) \in \dot{D}^\alpha(\vec{D})$.

(3) $\Gamma^\alpha_{\vec{\pi}}$ **is self-inverse modulo equivalence:** Let $p \in \mathcal{D}^\alpha(\vec{D})$. By the inductive assumption,

$$p \restriction \beta \leq_{\mathbb{P}_\beta} \Gamma^2(p \restriction \beta) \leq_{\mathbb{P}_\beta} p \restriction \beta$$

It follows that $\Gamma^\alpha_{\vec{\pi}}(p) \restriction \beta = \Gamma(p \restriction \beta)$ and $\Gamma^\alpha_{\vec{\pi}}(p)(\beta) = \Gamma^*(\dot{\pi}_\beta(\Gamma^*(\dot{\pi}_\beta(p(\beta)))))$. Note that $\Vdash \Gamma^*(\dot{\pi}_\beta(p(\beta))) = \Gamma^*(\dot{\pi}_\beta)(\Gamma^*(p(\beta)))$. Hence, by the assumptions placed on $\dot{\pi}_\beta$, we have that

$$\Vdash \dot{\pi}_\beta(\Gamma^*(\dot{\pi}_\beta(p(\beta)))) = \dot{\pi}_\beta(\Gamma^*(\dot{\pi}_\beta)(\Gamma^*(p(\beta)))) = \Gamma^*(p(\beta))$$

This clearly implies that $\Vdash \Gamma^*(\dot{\pi}_{\beta'}(\Gamma^*(\dot{\pi}_{\beta'}(p(\beta))))) = \Gamma^*(\Gamma^*(p(\beta))) = p(\beta)$ which implies the desired statement.

(4) $\Gamma^\alpha_{\vec{\pi}}$ **is an automorphism modulo equivalence:** This follows from the fact that $\Gamma^\alpha_{\vec{\pi}}$ is order-preserving and self-inverse modulo equivalence.

Case α is a limit: Suppose the statement holds for all $\beta < \alpha$.

(1) $\Gamma^\alpha_{\vec{\pi}}$ **preserves the orders:** Suppose that $p, p' \in \mathcal{D}^\alpha(\vec{D})$ and $p' \leq_{\mathbb{P}_\alpha} p$. Thus, for every $\beta < \alpha$, $p' \restriction \beta \Vdash p'(\beta) \leq_{\dot{\mathbb{Q}}_\beta} p(\beta)$ and there is $b \subseteq \mathrm{supp}(p')$ finite such that for all $\beta \in \mathrm{supp}(p') \setminus b$, $p' \restriction \beta \Vdash p'(\beta) \leq_{\dot{\mathbb{Q}}_\beta} p(\beta)$. It follows just as in the previous case that this implies that $\Gamma^\alpha_{\vec{\pi}}(p') \leq_{\mathbb{P}_\alpha} \Gamma^\alpha_{\vec{\pi}}(p)$. The same applies to $\leq^*_{\mathbb{P}_\alpha}$.

(2) $\Gamma^\alpha_{\vec{\pi}}$ **maps from $\mathcal{D}^\alpha(\vec{D})$ into $\mathcal{D}^\alpha(\vec{D})$:** We have that $p \in \mathcal{D}^\alpha(\vec{D})$ if and only if $p \in \mathbb{P}_\alpha$ and $p \restriction \beta \in \mathcal{D}^\beta(\vec{D})$. By Remark 4.2(2), we are only tasked with showing that $\Gamma^\alpha_{\vec{\pi}}(p) \in \mathbb{P}_\alpha$. But it is clear from the construction that $\mathrm{supp}(\Gamma^\alpha_{\vec{\pi}}(p)) = \mathrm{supp}(p)$, which implies the desired statement.

(3) $\Gamma^\alpha_{\vec{\pi}}$ **is self-inverse modulo equivalence:** This follows easily from the inductive assumption, using Remark 4.2(2).
(4) $\Gamma^\alpha_{\vec{\pi}}$ **is an automorphism modulo equivalence:** This again follows easily from the fact that $\Gamma^\alpha_{\vec{\pi}}$ is order-preserving and self-inverse modulo equivalence.

This finishes the proof. □

Since $\mathcal{D}(\vec{D})$ is $\leq^*$-dense in $\mathbb{P}_\rho$, Lemma 2.17 easily implies:

Corollary 4.4. *Let $(\dot\pi_\beta, \dot D_\beta)_{\beta<\alpha}$ be an automorphism sequence. Let G be a $\mathbb{P}_\alpha$-generic filter. Then the upward closure of $\Gamma_{\vec\pi}[G \cap \mathcal{D}(\vec D)]$ defines a $\mathbb{P}_\alpha$-generic filter.* □

Let us explain the construction behind the homogeneity. It is easy to see (and classical) that the usual Prikry-forcing (even tree-Prikry-forcing) is cone homogeneous: Given conditions $\langle t_0, T_0\rangle$ and $\langle t_1, T_1\rangle$, we take $T^* := (T_0 \cap T_1) \setminus \max\{\max(t_0), \max(t_1)\}$ and define an isomorphims γ between $\mathbb{Q}/\langle t_0, T^*\rangle$ and $\mathbb{Q}/\langle t_1, T^*\rangle$ taking $\gamma(\langle s, S\rangle) := \langle t_1{}^\frown(s \setminus \max(t_1)+1), S\rangle$.

However, in the case of the poset $\mathbb{Q}^\rho_\alpha$, it is not that simple: Given conditions $\langle t_0, T_0\rangle$ and $\langle t_1, T_1\rangle$, it is the case that $\mathrm{Suc}_{T_0}(t_0)$ is large for $\mathcal{U}_{\alpha,\rho}(t_0)$ and $\mathrm{Suc}_{T_1}(t_1)$ is large for $\mathcal{U}_{\alpha,\rho}(t_1)$. If $t_0 \neq t_1$, these two ultrafilters necessarily concentrate on disjoint sets: One concentrates on those ordinals whose Prikry/Magidor sequence begins with $t_0 \restriction \rho$, the other one concentrates on those ordinals whose Prikry/Magidor sequence begins with $t_1 \restriction \rho$. In fact, it can be shown that for ρ a limit ordinal of uncountable cofinality, $\mathbb{Q}^\rho_\alpha$ is not homogeneous.[16] However, here is where the prior iteration $\mathbb{P}_\alpha$ comes in: According to the definition of an automorphism sequence, given a $\mathbb{P}_\alpha$-generic filter G_α and letting $H_\alpha := \Gamma^\alpha_{\vec\pi}[G_\alpha]$, we are not aiming for an automorphism of $(\mathbb{Q}^\rho_\alpha)^{G_\alpha}$, but instead for an *isomorphism* of dense subsets of $(\mathbb{Q}^\rho_\alpha)^{G_\alpha}$ and $(\dot{\mathbb{Q}}^\rho_\alpha)^{H_\alpha}$.[17] Thus, by choosing our prior automorphism sequence correctly, for fixed conditions $\langle t_0, T_0\rangle$ and $\langle t_1, T_1\rangle$, we can make sure that, if $\langle t_0, T_0\rangle \in (\mathbb{Q}^\rho_\alpha)^{G_\alpha}$ and $\langle t_1, T_1\rangle \in (\dot{\mathbb{Q}}^\rho_\alpha)^{H_\alpha}$, T_0 splits according to the sequence of measures

$$\langle \dot{\mathcal{U}}^{H_\alpha}_{\alpha,\sigma}((t_1{}^\frown\vec\nu) \restriction \sigma) \mid \sigma < \rho\rangle$$

and vice versa, T_1 splits according to

$$\langle \dot{\mathcal{U}}^{G_\alpha}_{\alpha,\sigma}((t_0{}^\frown\vec\nu) \restriction \sigma) \mid \sigma < \rho\rangle.$$

This is achieved by swapping t_0 and t_1 *below* α so that whenever the Prikry/Magidor sequence of γ according to G_β begins with t_0, the Prikry sequence of γ according to H_β begins with t_1. Thanks to the fractal-like

[16] An easy way to show this is by using work of Hayut–Karagila [HK16]. Combining their Corollary 12 and Theorem 13, it follows that any weakly homogeneous poset changing the cofinality of κ to a regular cardinal $\mu > \omega$, without adding bounded subsets of κ, must change the cofinality of some $\lambda < \kappa$. Thus, if $\mathbb{Q}^\rho_\alpha$ were to be weakly homogeneous then it would have to change cofinalities below κ, which is not the case.

[17] It is easy to see that $(\Gamma^{\alpha,-1}_{\vec\pi})^*(\dot{\mathbb{Q}}^\rho_\alpha)^{G_\alpha} = (\dot{\mathbb{Q}}^\rho_\alpha)^{H_\alpha}$

structure of the poset, this amounts to the following task: We want to perform the same automorphism at α in V and at α in any $M_{\alpha,\rho}$. The easiest way to accomplish this is to perform an automorphism representing the one at α at a club of points below α. However, in any condition in $\mathbb{P}_{\kappa+1}$, we do not have to worry about just one condition, but unboundedly many. Fortunately, since the support of any condition is nowhere-stationary, we can partition the ordinals below κ into disjoint clubs for any α in the support of a given condition in such a way that this aligns nicely with the relevant ultrapowers. To ensure that this is possible, we prove a lemma giving us *coherent representations* for small sets which also align nicely with the relevant ultrapowers, allowing us to define an automorphism sequence such that, for every $(\alpha,\sigma)(\mathcal{U})$, the same automorphism is performed at α in $V[G_\alpha]$ and in $M_{\alpha,\sigma}[G_\alpha]$.

The first step is showing that nowhere-stationary sets admit disjoint ladder systems satisfying a strong zero-one law. Note that nowhere-stationarity here is crucial: If A where stationary in any regular cardinal, we could apply Fodor's lemma to $\alpha \mapsto \min(C_\alpha)$ which would show that the C_α cannot be pairwise disjoint.

Lemma 4.5. *Suppose that δ is a regular cardinal, $\gamma < \delta$ is an ordinal and $A \subseteq \delta$ is a nowhere-stationary set consisting of regular cardinals such that $A \cap (\gamma+1) = \emptyset$. Then, there exists a sequence $(C_\alpha)_{\alpha\in A}$ such that:*

1. *For every $\alpha \in A$, $C_\alpha \subseteq \alpha$ is club and $C_\alpha \cap \gamma = \emptyset$;*
2. *For every distinct $\alpha,\beta \in A$, $(C_\alpha \cup \{\alpha\}) \cap (C_\beta \cup \{\beta\}) = \emptyset$;*
3. *For every regular cardinal $\mu \leq \delta$, either $(\bigcup_{\alpha\in A} C_\alpha) \cap \mu$ is nonstationary in μ or there is $\alpha \in A$ such that $C_\alpha \cap \mu$ is club in μ. Moreover, for any $\alpha \in \delta$, if $\mu \in C_\alpha$ then $C_\alpha \cap \mu$ is club in μ.*

Proof. We prove the lemma by induction on δ.

Case $\delta = \theta^+$: Let $\gamma < \delta$ be an ordinal and $A \subseteq \delta$ a nowhere-stationary set of regular cardinals. If $\theta \notin A$ then the induction hypothesis hands us a sequence $(C_\alpha)_{\alpha\in A\cap\theta} = (C_\alpha)_{\alpha\in A}$ witnessing Clauses (1)–(3) above.

So, let us assume that $\theta \in A$. Let $C_\theta \subseteq \theta$ be a club set disjoint from $A \cup \gamma$. We may assume that for every $\gamma \in C_\theta$ with $\mathrm{cf}(\gamma) > \omega$, $C_\theta \cap \gamma$ is unbounded in γ. In particular, for every $\alpha \in A$, $C_\theta \cap \alpha$ is bounded in α. Fix a sequence $(D_\alpha)_{\alpha\in A\cap\theta}$ witnessing the inductive hypothesis for $A \cap \theta$ and γ. For $\alpha \in A \cap \theta$, let $C_\alpha := D_\alpha \setminus \sup(C_\alpha \cap \theta) + 1$.

Claim 4.5.1. *$(C_\alpha)_{\alpha\in A}$ satisfies the conclusion of the lemma.*

Proof of claim. Both (1) and and (2) hold by the inductive hypothesis and the definition of C_α. To see (3), let $\mu \leq \delta$ be a regular cardinal. Suppose that $(\bigcup_{\alpha\in A} C_\alpha) \cap \mu$ is stationary in μ. If $C_\theta \cap \mu$ is stationary in μ, we have that $C_\theta \cap \mu$ is club in μ – by assumption. So suppose that $\left(\bigcup_{\alpha\in A\cap\theta} C_\alpha\right) \cap \mu$ is stationary in μ. In particular, $\left(\bigcup_{\alpha\in A\cap\theta} D_\alpha\right) \cap \mu$ is stationary in μ. Additionally, we must have $\mu \leq \theta$. Therefore, there exists $\alpha \in A \cap \theta$ such

that $D_\alpha \cap \mu$ is club in μ. Note that D_α witnesses that $(\bigcup_{\beta\in(A\cap\theta)\setminus\{\alpha\}} D_\alpha)\cap\mu$ and therefore also

$$\left(\bigcup_{\beta\in(A\cap\theta)\setminus\{\alpha\}} C_\alpha\right)\cap\mu$$

is nonstationary in μ. Thus, $C_\alpha\cap\mu$ must be stationary in μ, since we were assuming that $(\left(\bigcup_{\beta\in(A\cap\theta)\setminus\{\alpha\}} C_\beta\right)\cup C_\alpha)\cap\mu$ is stationary in μ. But this implies that $\mu\in C_\alpha$ and so $C_\alpha\cap\mu$ is actually club in μ. □

Case δ is a limit: As before, find $C\subseteq\delta$ club such that $C\cap A=\emptyset$. Enumerate $C=\langle\delta_i\mid i<\delta\rangle$, where $\gamma=\delta_0$. Thus, $\delta_i\notin A$ for all $i<\delta$. For each $i<\delta$, apply the inductive hypothesis to $A\cap(\delta_i,\delta_{i+1})$ and δ_i to find a sequence $(C_\alpha)_{\alpha\in A\cap(\delta_i,\delta_{i+1})}$ witnessing the inductive hypothesis for $A\cap(\delta_i,\delta_{i+1})$ and δ_i, respectively. Let us show that $(C_\alpha)_{\alpha\in A}$ fulfills (1)–(3).

Clause (1) is clear. For Clause (2), let $\alpha<\beta$ be elements of A. If there is $i<\delta$ such that $\alpha,\beta\in(\delta_i,\delta_{i+1})$ then $(C_\alpha\cup\{\alpha\})\cap(C_\beta\cup\{\beta\})=\emptyset$ because of the properties of the sequence $(C_\alpha)_{\alpha\in A\cap(\delta_i,\delta_{i+1})}$. Otherwise, there is $i<\delta$ such that $\alpha<\delta_i<\beta$. In this case, we have $C_\alpha\subseteq\alpha<\delta_i$ and $C_\beta\cap\delta_i=\emptyset$, so

$$(C_\alpha\cup\{\alpha\})\cap(C_\beta\cup\{\beta\})=\emptyset.$$

For (3), let $\mu\leq\delta$ be a regular cardinal. For $\mu=\delta$, clearly, C is club in δ and disjoint from $\bigcup_{\alpha\in A} C_\alpha$, ergo the latter is nonstationary. So, suppose that $\mu<\delta$. If $C\cap\mu$ is unbounded in μ, it must be club in μ and, again, it is disjoint from $(\bigcup_{\alpha\in A} C_\alpha)\cap\mu$. Otherwise, there is $i<\delta$ such that $\delta_i<\mu<\delta_{i+1}$. If $(\bigcup_{\alpha\in A} C_\alpha)\cap\mu$ is stationary in μ, $\bigcup_{\alpha\in A\cap(\delta_i,\delta_{i+1})} C_\alpha$ must be stationary in μ as well, for $\bigcup_{\alpha\in A\cap\delta_i} C_\alpha\subseteq\delta_i<\mu$ and $(\bigcup_{\alpha\in A\cap(\delta_{i+1},\delta)} C_\alpha)\cap\delta_{i+1}=\emptyset$. It follows from the inductive hypothesis that there is $\alpha\in A\cap(\delta_i,\delta_{i+1})$ such that $C_\alpha\cap\mu$ is club in μ, as sought. □

The second ingredient is the following lemma, stating that, for $\alpha\in\mathrm{dom}_1(\mathcal{U})$, sets in $V_{\alpha+1}$ can be represented using functions which cohere with respect to the ultrapowers according to $\mathcal{U}_{\beta,\sigma}$ for $\beta\leq\alpha$.

Lemma 4.6. *Suppose $\alpha\in\mathrm{dom}_1(\mathcal{U})$ and $x\in V_{\alpha+1}$. Then, there is a function $h\colon\alpha\to V$ such that the following holds:*

(1) For every $\sigma<o^{\mathcal{U}}(\alpha)$, $j_{\alpha,\sigma}(h)(\alpha)=x$;
(2) For every $(\beta,\sigma)\in\mathrm{dom}(\mathcal{U})$, $j_{\beta,\sigma}(h)(\beta)=h(\beta)$.

Proof. Since $x\in V_{\alpha+1}\subseteq M_{\alpha,0}$, there is a function $f\colon\alpha\to V_\alpha$ such that

$$j_{\alpha,0}(f)(\alpha)=x.$$

Let $g\colon\alpha\to V$ denote the following function:

$$g(\beta):=\begin{cases}\emptyset, & o^{\mathcal{U}}(\beta)=0;\\ j_{\beta,0}(f\restriction\beta)(\beta), & o^{\mathcal{U}}(\beta)>0.\end{cases}$$

Let us show that g satisfies the first requirement:

Claim 4.6.1. *For every* $0 < \sigma < o^{\mathcal{U}}(\alpha)$,

$$j_{\alpha,\sigma}(g)(\alpha) = x.$$

Proof. By coherency, $j_{\alpha,\sigma}(\mathcal{U})_{\alpha,0} = \mathcal{U}_{\alpha,0}$. Ergo,

$$(j_{j_{\alpha,\sigma}(\mathcal{U})_{\alpha,0}})^{M_{\alpha,\sigma}} = (j_{\mathcal{U}_{\alpha,0}})^{M_{\alpha,\sigma}} = j_{\alpha,0} \restriction M_{\alpha,\sigma}.$$

Therefore, we have

$$j_{\alpha,\sigma}(g)(\alpha) = (j_{j_{\alpha,\sigma}(\mathcal{U})_{\alpha,0}})^{M_{\alpha,\sigma}}(j_{\alpha,\sigma}(f) \restriction \alpha)(\alpha) = j_{\alpha,0}(f)(\alpha) = x.$$

The second of these equalities follows from two observations: First, we have $j_{\alpha,\sigma}(f) \restriction \alpha = f$ as $\operatorname{range}(f) \subseteq V_\alpha$ and $\operatorname{crit}(j_{\alpha,\sigma}) = \alpha$. Second, the function $f\colon \alpha \to V_\alpha$ belongs to $M_{\alpha,\sigma}$ (in that this ultrapower is closed under α-sequences of its members) and so $(j_{j_{\alpha,\sigma}(\mathcal{U})_{\alpha,0}})^{M_{\alpha,\sigma}}(f) = j_{\alpha,0}(f)$. □

Claim 4.6.2. *For every* $\beta < \alpha$ *with* $o^{\mathcal{U}}(\beta) > 0$ *and every* $0 < \sigma < o^{\mathcal{U}}(\beta)$,

$$j_{\beta,\sigma}(g \restriction \beta)(\beta) = j_{\beta,0}(f \restriction \beta)(\beta).$$

Proof. By the same argument as in the previous claim,

$$j_{\beta,\sigma}(g \restriction \beta)(\beta) = (j_{j_{\beta,\sigma}(\mathcal{U})_{\beta,0}})^{M_{\beta,\sigma}}(j_{\beta,\sigma}(f \restriction \beta) \restriction \beta) = j_{\beta,0}(f \restriction \beta)(\beta).$$

□

So we can let $h\colon \alpha \to V$ be a function defined as

$$h(\beta) := \begin{cases} f(\beta), & o^{\mathcal{U}}(\beta) = 0; \\ g(\beta), & o^{\mathcal{U}}(\beta) > 0. \end{cases}$$

Now we can show that h is as required by proving the following two claims:

Claim 4.6.3. *For every* $\sigma < o^{\mathcal{U}}(\alpha)$, $j_{\alpha,\sigma}(h)(\alpha) = x$.

Proof. If $\sigma = 0$ then $j_{\alpha,0}(h)(\alpha) = j_{\alpha,0}(f)(\alpha) = x$ (because, by coherency, $o^{j_{\alpha,0}(\mathcal{U})}(\alpha) = 0$). Similarly, for $\sigma > 0$, $j_{\alpha,\sigma}(h)(\alpha) = j_{\alpha,\sigma}(g)(\alpha) = x$. □

Claim 4.6.4. *For every* $\beta < \alpha$ *with* $o^{\mathcal{U}}(\beta) > 0$ *and every* $\sigma < o^{\mathcal{U}}(\beta)$,

$$j_{\beta,\sigma}(h \restriction \beta)(\beta) = h(\beta).$$

Proof. For $o^{\mathcal{U}}(\beta) > 0$ and $\sigma < o^{\mathcal{U}}(\beta)$, the claim follows from the definition of h and because $o^{j_{\beta,0}(\mathcal{U})}(\beta) = 0$. Indeed:

$$h(\beta) = g(\beta) = j_{\beta,0}(f \restriction \beta)(\beta) = j_{\beta,0}(h \restriction \beta)(\beta).$$

Suppose $\sigma > 0$, then combining the Claim 4.6.2 with the definition of h:

$$j_{\beta,\sigma}(h \restriction \beta)(\beta) = j_{\beta,\sigma}(g \restriction \beta)(\beta) = j_{\beta,0}(f \restriction \beta)(\beta) = g(\beta) = h(\beta).$$

This completes the proof of the claim. □

We are done with the proof of the lemma. □

To finish the preparations, we combine the previous two lemmas to show that we can represent sequences indexed by nowhere-stationary subsets of $\mathrm{dom}_1(\mathcal{U})$ in a way that coheres with the ultrapowers by *every* measure $\mathcal{U}_{\beta,\sigma}$ where $(\beta,\sigma)\in\mathrm{dom}(\mathcal{U})$.

Lemma 4.7. *Let $A\subseteq \mathrm{dom}_1(\mathcal{U})$ be nowhere-stationary. Let $(t_\alpha)_{\alpha\in A}$ be a sequence such that for every $\alpha\in A$, $t_\alpha\in V_{\alpha+1}$. Then there is a function*

$$h\colon \kappa+1\to V_{\kappa+1}$$

such that the following holds:

(1) For every $\alpha\in A$, $h(\alpha)=t_\alpha$;
(2) For every $\beta\in\mathrm{dom}_1(\mathcal{U})$ and $\sigma\leq o^{\mathcal{U}}(\beta)$, $j_{\beta,\sigma}(h)(\beta)=h(\beta)$.

Proof. Fix a sequence $(C_\alpha)_{\alpha\in A}$ exemplifying Lemma 4.5, i.e. for every $\alpha\in A$, $C_\alpha\subseteq\alpha$ is club, for every distinct $\alpha,\beta\in A$, $(C_\alpha\cup\{\alpha\})\cap(C_\beta\cup\{\beta\})=\emptyset$ and for every regular cardinal $\mu\leq\kappa$, either $\left(\bigcup_{\alpha\in A}C_\alpha\right)\cap\mu$ is nonstationary in μ or there is $\alpha\in A$ such that $C_\alpha\cap\mu$ is club in μ. Moreover, for any $\alpha\in\delta$, if $\mu\in C_\alpha$, then $C_\alpha\cap\mu$ is club in μ.

For every $\beta\in\kappa+1$, let $g(\beta)$ be the unique α with $\beta\in C_\alpha\cup\{\alpha\}$ if it exists, 0 otherwise. Also fix, for every $\alpha\in A$, a function $f_\alpha\colon\kappa+1\to V_{\kappa+1}$ such that $f_\alpha(\alpha)=t_\alpha$ and for every $\beta\in\mathrm{dom}_1(\mathcal{U})$ and $\sigma<o^{\mathcal{U}}(\beta)$,

$$j_{\beta,\sigma}(f_\alpha)(\beta)=f_\alpha(\beta)$$

This is possible by Lemma 4.6: Let $f'_\alpha\colon\alpha\to V_{\alpha+1}$ be a function exemplifying Lemma 4.6 for $x=t_\alpha$. Define $f_\alpha\colon\kappa+1\to V_{\alpha+1}$ by letting $f_\alpha(\alpha):=t_\alpha$ and $f_\alpha\restriction(\alpha,\kappa+1)\equiv\emptyset$. For $\beta\in\mathrm{dom}_1(\mathcal{U})\cap\alpha$ and $\sigma<o^{\mathcal{U}}(\beta)$, we have

$$j_{\beta,\sigma}(f_\alpha)(\beta)=(j_{\alpha,\sigma}(f_\alpha)\restriction j_{\beta,\sigma}(\alpha))=j_{\beta,\sigma}(f_\alpha\restriction\alpha)(\beta)=j_{\beta,\sigma}(f'_\alpha)(\beta)=f'_\alpha(\beta)=f_\alpha(\beta).$$

For $\beta=\alpha$ and $\sigma<o^{\mathcal{U}}(\beta)$, we have

$$j_{\beta,\sigma}(f_\alpha)(\beta)=t_\alpha=f_\alpha(\alpha)$$

For $\beta\in\mathrm{dom}_1(\mathcal{U})\cap(\alpha,\kappa+1)$, we have

$$j_{\beta,\sigma}(f_\alpha)(\beta)=\emptyset=f_\alpha(\beta)$$

where the first equality follows from the fact that $j_{\beta,\sigma}(\alpha)<j_{\beta,\sigma}(\beta)$ by elementarity.

Now we define

$$h(\beta):=\begin{cases}f_{g(\beta)}(\beta), & g(\beta)\neq 0;\\ \emptyset, & g(\beta)=0.\end{cases}$$

Let us show that this works. Clearly, for $\alpha\in A$,

$$h(\alpha)=f_{g(\alpha)}(\alpha)=f_\alpha(\alpha)=t_\alpha.$$

Now suppose that $(\beta,\sigma)\in\mathrm{dom}(\mathcal{U})$.

First suppose that $g(\beta) = 0$. Thus, $\left(\bigcup_{\alpha\in A} C_\alpha\right) \cap \beta$ is nonstationary in β and so, since $A \cap \beta$ is nonstationary in β as well,

$$\beta \notin j_{\beta,\sigma}\left(\bigcup_{\alpha\in A} C_\alpha \cup \{\alpha\}\right)$$

Therefore, $j_{\beta,\sigma}(g)(\beta) = 0$, implying $j_{\beta,\sigma}(h)(\beta) = 0 = h(\beta)$.

Suppose now that $g(\beta) = \alpha \neq 0$. Therefore, $C_\alpha \cap \beta$ is club in β and thus $\beta \in j_{\beta,\sigma}(C_\alpha)$. Since $j_{\beta,\sigma}(C_\alpha) = j_{\beta,\sigma}((C_\alpha)_{\alpha\in A})(j_{\beta,\sigma}(\alpha))$, this shows $j_{\beta,\sigma}(g)(\beta) = j_{\beta,\sigma}(\alpha)$. Now we have

$$j_{\beta,\sigma}(h)(\beta) = j_{\beta,\sigma}((f_\alpha)_{\alpha\in A})_{(j_{\beta,\sigma}(\alpha))}(\beta) = j_{\beta,\sigma}(f_\alpha)(\beta) = f_\alpha(\beta) = h(\beta)$$

by assumption. □

Now that we are done with the preparations, we can move on to the prove of the main result of this section. For the purposes of this proof, it will be helpful to restrict ourselves to sequences of a specific type:

Definition 4.8. Let $(\beta, \tau) \in \mathrm{dom}(\mathcal{U})$ and work in $V[G_\beta]$, where G_β is $\mathbb{P}_\beta$-generic. Let $x \in (\beta)^{<\omega}$. We say that x is a *minimal τ-coherent sequence* if x is τ-coherent and for all $i < \mathrm{dom}(x) - 1$, $o^{\mathcal{U}}(x(i)) \geq o^{\mathcal{U}}(x(i+1))$.

We call such a sequence *minimal* because it has minimal length among all sequences which are equivalent to it (in the sense that they induce the same Prikry sequence).

Definition 4.9. Let $(\beta, \tau) \in \mathrm{dom}(\mathcal{U})$ and work in $V[G_\beta]$, where G_β is $\mathbb{P}_\beta$-generic. Let x. We let $\mathrm{reg}(x)$ be the subsequence of x consisting of all $x(i)$ such that $o^{\mathcal{U}}(x(i)) \geq o^{\mathcal{U}}(x(j))$ for all $j > i$.

It is easy to see that for every τ-coherent sequence x, $\mathrm{reg}(x)$ is the unique minimal τ-coherent sequence which is equivalent to x.

Theorem 4.10. *$\mathbb{P}_{\kappa+1}$ is almost homogeneous modulo equivalence.*

Proof. Let $q_0, q_1 \in \mathbb{P}_{\kappa+1}$. To simplify the notation, we may assume without loss of generality that $\mathrm{supp}(q_0) = \mathrm{supp}(q_1)$. Denote this common set by A.

Our goal is to find the following:

(1) An automorphism sequence $(\vec{\pi}, \vec{D}) = (\dot{\pi}_\beta, \dot{D}_\beta)_{\beta<\kappa+1}$;
(2) Pure extensions $w_0 \leq^* q_0$, $w_1 \leq^* q_1$ in $\mathbb{P}_{\kappa+1}$ such that $w_0, w_1 \in \mathcal{D}(\vec{D})$ and $\Gamma_{\vec{\pi}}(w_0) = w_1$.

Let $\alpha \in A$. If $o(\alpha) < \alpha^+$, let $\dot{t}^i_\alpha$ be such that $q_i(\alpha) = \langle \dot{t}^i_\alpha, \dot{T}^i_\alpha \rangle$. If $o^{\mathcal{U}}(\alpha) \geq \alpha^+$, let $\dot{t}^i_\alpha$ be such that $q_i(\alpha) = \langle \langle \dot{t}^i_\alpha, \dot{T}^i_\alpha \rangle, \dot{r}^i_\alpha \rangle$. By Lemma 4.7, fix functions $h_0, h_1 \colon \kappa + 1 \to V_{\kappa+1}$ such that, for every $\alpha \in A$, $h_i(\alpha) = \dot{t}^i_\alpha$ and for every $(\beta, \sigma) \in \mathrm{dom}(\mathcal{U})$,

$$j_{\beta,\sigma}(h_i)(\beta) = h_i(\beta)$$

We construct the sequence $(\dot{\pi}_\beta, \dot{D}_\beta)_{\beta<\kappa+1}$ by induction on β. For now, simply suppose that $(\dot{\pi}_\gamma, \dot{D}_\gamma)_{\gamma<\beta}$ has been constructed and is an automorphism sequence. If $\beta \notin \mathrm{dom}_1(\mathcal{U})$, there is nothing to do, since the βth iterand $\dot{\mathbb{Q}}_\beta$ is forced to be trivial. So, moving forward we assume that $\beta \in \mathrm{dom}_1(\mathcal{U})$. To describe the inductive hypotheses, work in $V[G_\beta]$, where G_β is $\mathbb{P}_\beta$-generic. Define H_β to be the upward closure of $\Gamma_{\vec{\pi}}[H_\beta]$, which is $\mathbb{P}_\beta$-generic as well, by Corollary 4.4.

A central notion is the following:

Definition 4.11. Let t be a finite sequence of ordinals. Let $K_\beta \subseteq \mathbb{P}_\beta$ be a generic filter. Let $\tau \leq o^{\mathcal{U}}(\beta)$. We say that t is *τ-coherent with respect to K_β* if it is a τ-coherent sequence in $V[K_\beta]$ according to Definition 3.2. We say that t is *coherent with respect to K_β* if it is $o^{\mathcal{U}}(\beta)$-coherent with respect to K_β.

For every $\tau \leq o^{\mathcal{U}}(\beta)$, we define an auxiliary function σ^τ_β as follows:

Definition 4.12. σ^τ_β is a function on the set of all minimal τ-coherent sequences of ordinals below β. The values of σ^τ_β are defined as follows:

Let us denote

$$s_0^G := (h_0(\beta))^{G_\beta},\ s_0^H := (h_0(\beta))^{H_\beta} \text{ and } s_1^G := (h_1(\beta))^{G_\beta},\ s_1^H := (h_1(\beta))^{H_\beta}$$

(1) If there exists r such that $(s_0^G \restriction \tau)^\frown r$ is τ-coherent with respect to G_β, $(s_1^H \restriction \tau)^\frown r$ is τ-coherent with respect to H_β and $t = \mathrm{reg}((s_0^G \restriction \tau)^\frown r)$, then
$$\sigma_\beta(t) = \mathrm{reg}((s_1^H \restriction \tau)^\frown r).$$
(2) If there exists r such that $(s_1^G \restriction \tau)^\frown r$ is τ-coherent with respect to G_β, $(s_0^H \restriction \tau)^\frown r$ is τ-coherent with respect to H_β and $t = \mathrm{reg}((s_1^G \restriction \tau)^\frown r)$, then
$$\sigma_\beta(t) = \mathrm{reg}((s_0^H \restriction \tau)^\frown r).$$
(3) Otherwise, $\sigma_\beta(t) = t$.

Remark 4.13. The idea behind the map σ^τ_β is as follows: At stage α, σ^τ_α is going to swap $h_0(\alpha) \restriction \tau$ and $h_1(\alpha) \restriction \tau$, i.e. $\dot{t}^0_\alpha \restriction \tau$ and $\dot{t}^1_\alpha \restriction \tau$. So that this is possible, we must do the same procedure at many places below α as well.

Ordinarily, we would want to simply map $s_0^\frown r$ to $s_1^\frown r$ and vice versa, ignoring sequences which do not extend s_0 or s_1. However, due to the intricate definition of the forcing, there can be sequences which do not extend s_0 or s_1 per se, but are equivalent to sequences extending s_0 or s_1. So that we preserve the ordering, we must also swap sequences like this and therefore deal with equivalence of sequences.

Remark 4.14. Note that the definition of the map reg is independent of the choice of generic (assuming the sequence is coherent with respect to the correct one), since it simply gets rid of all ordinals which are superceded by an ordinal of larger order, which is decided in the ground model.

We formulate our **induction hypothesis** in V. We assume the following:

- (ℵ) $(\vec{\pi}, \vec{D}) = (\dot{\pi}_\gamma, \dot{D}_\gamma)_{\gamma<\beta}$ is an automorphism sequence.
- (⊐) For every $\gamma < \beta$ with $o(\gamma) < \gamma^+$, for every $\mathbb{P}_\gamma$-generic filter G_γ, setting $\pi_\gamma := (\dot{\pi}_\gamma)^{G_\gamma}$ and $D_\gamma := (\dot{D}_\gamma)^{G_\gamma}$, it is the case that
$$\pi_\gamma \colon D_\gamma \to (\dot{D}_\gamma)^{H_\gamma}$$[18]
is an isomorphism that is defined as
$$\pi_\gamma(\langle t, T\rangle) := \langle \sigma_\gamma^{o^{\mathcal{U}}(\gamma)}(t), T\rangle.$$
For every $\gamma \in \mathrm{dom}_1(\mathcal{U}) \cap \beta$ with $o^{\mathcal{U}}(\gamma) = \gamma^+$, for every $\mathbb{P}_\gamma$-generic filter G_γ, setting $\pi_\gamma := (\dot{\pi}_\gamma)_{G_\gamma}$ and $D_\gamma := (\dot{D}_\gamma)_{G_\gamma}$, the following holds: If $\gamma \notin A$, then
$$D_\gamma = \mathbb{Q}_\gamma \text{ and } \pi_\gamma = \mathrm{id}_{\mathbb{Q}_\gamma}$$
If $\gamma \in A$, then there is a $\leq^*_\gamma$-dense set $E_\gamma \subseteq \mathbb{Q}_\gamma^{\gamma^+}$, an isomorphism τ_γ from E_γ to $(\dot{E}_\gamma)^{H_\gamma}$ and a self-inverse bijection ι_γ between γ^+ and γ^+ such that
$$\Vdash_{\mathbb{Q}_\gamma^{\gamma^+}} \iota_\gamma[\mathrm{supp}(\dot{r}^i_\gamma)] \cap \mathrm{supp}(\dot{r}^i_\gamma) = \emptyset$$[19]
and
$$\tau_\gamma(\langle t, T\rangle) = \langle \sigma_\gamma^{\gamma^+}(t), T\rangle$$
as well as
$$\pi_\gamma(\langle t, T\rangle, \dot{r}) = (\tau_\gamma(\langle t, T\rangle), \tau^*_\gamma(\dot{\sigma}_{\iota_\gamma}(\dot{r})))$$
Here, $\dot{\sigma}_{\iota_\gamma}$ is a $\mathbb{Q}_\gamma^{\gamma^+}$-name for the automorphism of $\dot{\mathbb{R}}_\gamma$ that maps $\dot{r}$ to the unique condition with $\mathrm{supp}(\dot{\sigma}_{\iota_\gamma}(\dot{r})) = \check{\iota}_\gamma[\mathrm{supp}(\dot{r})]$ such that $\dot{\sigma}_{\iota_\gamma}(\dot{r})(\check{\iota}_\gamma(\xi) = \dot{r}(\xi)$.

The main goal for the rest of the proof is showing that, on a $\leq^*_\beta$-dense subset of $\mathbb{Q}_\beta := (\dot{\mathbb{Q}}_\beta)^{G_\beta}$, the sequence $\langle \sigma^\tau_\beta \mid \tau \leq o^{\mathcal{U}}(\beta)\rangle$ is well-defined and that it leads to a longer automorphism sequence $(\dot{\pi}_\gamma, \dot{D}_\gamma)_{\gamma<\beta+1}$ satisfying Clauses (ℵ) and (⊐).

To simplify the notation, we fix the following:

Notation 4.15. Let t be a finite sequence of ordinals below β. If t is τ-coherent with respect to G_β, let $\mathcal{U}_{\beta,\tau}(t)$ be the lift of $\mathcal{U}_{\beta,\tau}$ according to G_β. If t is τ-coherent with respect to H_β, let $\mathcal{V}_{\beta,\tau}(t)$ be the lift of $\mathcal{U}_{\beta,\tau}$ according to H_β.

The crucial claim is the following:

Claim 4.15.1. *Let t be a finite sequence of ordinals below β and $\tau \leq o^{\mathcal{U}}(\beta)$. Suppose that that t τ-coherent with respect to G_β and $\sigma^\tau_\beta(t)$ is τ-coherent with respect to H_β. Then $\mathcal{U}_{\beta,\tau}(t) = \mathcal{V}_{\beta,\tau}(\sigma^\tau_\beta(t))$.*

[18] Recall that $(\dot{D}_\gamma)^{H_\gamma} = (\Gamma^{\gamma,-1}_{\vec{\pi}})^*(\dot{D}_\gamma)^{G_\gamma}$

[19] Here we regard $\dot{r}^i_\gamma$ as a $\mathbb{Q}_\gamma^{\gamma^+}$-name inside $V[G_\gamma]$

Remark 4.16. Technically, $\sigma^\tau_\beta(t)$ is only defined when t is minimal, i.e. order-decreasing. However, we may and will ignore this distinction by letting $\sigma^\tau_\beta(t) := \sigma^\tau_\beta(\mathrm{reg}(t))$ if t is τ-coherent with respect to G_β.

Proof of Claim 4.15.1. Suppose that $X \in \mathcal{U}_{\beta,\tau}(t)$. Let $\dot{X}$ be a $\mathbb{P}_\beta$-name such that $\dot{X}_{G_\beta} = X$. By the definition of $\mathcal{U}_{\beta,\tau}(t)$, there exist $p \in G_\beta$ and a $\mathbb{P}_\beta$-name $\dot{T}$ such that

$$p^\frown\langle t, \dot{T}\rangle^\frown(j_{\beta,\tau}(p) \setminus (\beta+1)) \Vdash \check{\beta} \in j_{\beta,\tau}(\dot{X})$$

Since $\mathcal{D}(\vec{D})$ is $\leq^*$-dense, we may find $q \leq_{\mathbb{P}_\beta} p$ such that $q \in G_\beta \cap \mathcal{D}(\vec{D})$. Furthermore, we can find such a q such that it decides the values of $(h_0(\beta))$, $(h_1(\beta))$ and their translates by $\Gamma_{\vec{\pi}}$. Therefore, $j_{\beta,\tau}(q) \in j_{\beta,\tau}(\mathcal{D}(\vec{D})) = \mathcal{D}(j_{\beta,\tau}(\vec{D}))$. Thus, for every $\gamma \in \mathrm{supp}(j_{\beta,\tau}(q))$, $j_{\beta,\tau}(q) \restriction \gamma \Vdash j_{\beta,\tau}(q)(\gamma) \in j_{\beta,\tau}(\vec{D})(\gamma)$. By the maximum principle, let $\langle t, \dot{T}'\rangle$ be a $\mathbb{P}_\beta$-name for a pure extension of $\langle t, \dot{T}\rangle$ that lies in $j_{\beta,\tau}(\vec{D})(\beta)$. It follows that

$$q^\frown\langle t, \dot{T}'\rangle^\frown(j_{\beta,\tau}(q) \setminus \beta+1) \in j_{\beta,\tau}(\mathcal{D}(\vec{D}))$$

Therefore, we may assume that already $p^\frown\langle t, \dot{T}'\rangle^\frown(j_{\beta,\tau}(p) \setminus (\beta+1)) \in j_{\beta,\tau}(\mathcal{D}(\vec{D}))$. By the inductive hypothesis, $\Gamma_{j_{\beta,\tau}(\vec{\pi})}$ is an almost automorphism modulo equivalence of the $\leq^*$-dense set $\mathcal{D}(j_{\beta,\tau}(\vec{D}))$ (see Lemma 4.3). Therefore, we have

$$\Gamma_{j_{\beta,\tau}(\vec{\pi})}\left(p^\frown\langle t, \dot{T}\rangle^\frown(j_{\beta,\tau}(p) \setminus (\beta+1))\right) \Vdash \check{\beta} \in \Gamma^*_{j_{\beta,\tau}(\vec{\pi})}(j_{\beta,\sigma}(\dot{X})).$$

The following is clear, by elementarity and the fact that $j_{\beta,\tau}$ has critical point β:

(1) $\Gamma_{j_{\beta,\tau}(\vec{\pi})} = j_{\beta,\tau}(\Gamma_{\vec{\pi}})$;
(2) $\Gamma_{j_{\beta,\tau}(\vec{\pi})} \restriction \mathbb{P}_\beta = \Gamma_{\vec{\pi}}$.

Now consider $j_{\beta,\tau}(\vec{\pi})(\beta)$. Work in $M_{\sigma,\tau}$ for now. By Clause (ב) in the inductive hypothesis on page 39 and the elementarity of $j_{\beta,\tau}$, it is forced that

$$j_{\beta,\tau}(\vec{\pi})(\beta)(\langle \check{t}, \dot{T}\rangle) = \langle j_{\beta,\tau}(\vec{\sigma})^{o^{j_{\beta,\tau}(\mathcal{U})}(\beta)}_\beta(\check{t}), \dot{T}\rangle.$$

Note that, in $M_{\beta,\tau}$, $o^{j_{\beta,\tau}(\mathcal{U})}(\beta) = \tau$. Here, $j_{\beta,\tau}(\vec{\sigma})^\tau_\beta$ is a $\mathbb{P}_\beta$-name. This name is defined from the $\mathbb{P}_\beta$-names $j_{\beta,\tau}(h_0)(\beta)$, $j_{\beta,\tau}(h_1)(\beta)$ and their $\Gamma_{\vec{\pi}}$-translates. By assumption, we have $j_{\beta,\tau}(h_0)(\beta) = h_0(\beta)$ and $j_{\beta,\tau}(h_1)(\beta) = h_1(\beta)$ and, by assumption, all of the relevant names are decided by p. Ergo, $j_{\beta,\tau}(\vec{\sigma})^\tau_\beta$ as computed in $M_{\beta,\tau}[G_\beta]$ coincides with σ^τ_β as computed in $V[G_\beta]$. Let $s := \sigma^\tau_\beta(t)$, computed in $V[G_\beta]$. It follows that

$$p \Vdash j_{\beta,\tau}(\vec{\pi})(\beta)(\langle t, \dot{T}\rangle) = \langle \check{s}, \dot{T}\rangle$$

Therefore, working again in $V[G_\beta]$, we have:

$$\begin{aligned}
&\Gamma_{j_{\beta,\tau}(\vec{\pi})}\left(p^\frown\langle t,\dot{T}\rangle^\frown(j_{\beta,\tau}(p)\smallsetminus\beta+1)\right)\\
&=\Gamma_{\vec{\pi}}(p)^\frown\Gamma^*_{\vec{\pi}}(j_{\beta,\sigma}(\vec{\pi})(\beta)(\langle t,\dot{T}\rangle))^\frown(j_{\beta,\tau}\left(\Gamma_{\vec{\pi}}(p)\right)\smallsetminus(\beta+1))\\
&=\Gamma_{\vec{\pi}}(p)^\frown\Gamma^*_{\vec{\pi}}(\langle\check{s},\dot{T}\rangle))^\frown(j_{\beta,\tau}\left(\Gamma_{\vec{\pi}}(p)\right)\smallsetminus(\beta+1))\\
&=\Gamma_{\vec{\pi}}(p)^\frown\langle\check{s},\Gamma^*_{\vec{\pi}}(\dot{T})\rangle)^\frown(j_{\beta,\tau}\left(\Gamma_{\vec{\pi}}(p)\right)\smallsetminus(\beta+1)).
\end{aligned}$$

So $\Gamma_{\vec{\pi}}(p)\in H_\beta$ and $\Gamma^*_{\vec{\pi}}(\dot{T})$ witness that $\Gamma^*_{\vec{\pi}}(\dot{X})^{H_\beta}=X\in\mathcal{V}_{\beta,\sigma}(\sigma^\tau_\beta(t))$. □

We are now ready to define our $\leq^*_{\mathbb{Q}_\beta}$- dense subset D_β.

Let $\langle t,T\rangle\in\mathbb{Q}_\beta$ be arbitrary, and we define when $\langle t,T\rangle\in D_\beta$. For $\langle t,T\rangle\in D_\beta$, we require first of all that t is a minimal $o^{\mathcal{U}}(\beta)$-coherent sequence.

Recall that

$$s^G_0:=(h_0(\beta))^{G_\beta},\ s^H_0:=(h_0(\beta))^{H_\beta}\text{ and }s^G_1:=(h_1(\beta))^{G_\beta},\ s^H_1:=(h_1(\beta))^{H_\beta}$$

(see Definition 4.12). Let

$$\Xi:=\max(\max(s^G_0),\max(s^H_0),\max(s^G_1),\max(s^H_1))$$

(1) **Case A:** Suppose that, for some finite sequence $r\in\beta^{<\omega}$, $(s^G_0)^\frown r$ is $o^{\mathcal{U}}(\beta)$-coherent with respect to G_β, $(s^H_1)^\frown r$ is $o^{\mathcal{U}}(\beta)$-coherent with respect to H_β and $t=\mathrm{reg}((s^G_0)^\frown r)$. We let $\langle t,T\rangle\in D_\beta$ if and only if for every $s\in T$, $\mathrm{reg}((s^H_1)^\frown r)^\frown s$ is coherent with respect to H_β and $\min(s)>\Xi$.

(2) **Case B:** Suppose that, for some finite sequence $r\in\beta^{<\omega}$, $(s^G_1)^\frown r$ is $o^{\mathcal{U}}(\beta)$-coherent with respect to G_β, $(s^H_0)^\frown r$ is $o^{\mathcal{U}}(\beta)$-coherent with respect to H_β and $t=\mathrm{reg}((s^G_1)^\frown r)$. We let $\langle t,T\rangle\in D_\beta$ if and only if for every $s\in T$, $\mathrm{reg}((s^H_0)^\frown r)^\frown s$ is coherent with respect to G_β and $\min(s)>\Xi$.

(3) **Case C:** Suppose otherwise. We let $\langle t,T\rangle\in D_\beta$ if and only if for every $s\in T$, $\min(s)>\Xi$ and $t^\frown s$ is coherent with respect to both G_β and H_β.

Our goal for now is to show:

Claim 4.16.1. *D_β is $\leq^*_{\mathbb{Q}_\beta}$-open dense in $\mathbb{Q}_\beta$.*

Proof. Let $\langle t,T\rangle\in\mathbb{Q}_\beta$. By replacing t with $\mathrm{reg}(t)$, which results in a pure extension, we may suppose that t is minimal. We define the set X_t as follows:

(1) Suppose that **Case A** holds, witnessed by r. We let $\xi\in X_t$ if and only if $\mathrm{reg}((s^G_0)^\frown r)^\frown\langle\xi\rangle$ is coherent with respect to G_β and $\mathrm{reg}((s^H_1)^\frown r)^\frown\langle\xi\rangle$ is coherent with respect to H_β and $\xi>\Xi$.

(2) Suppose that **Case B** holds, witnessed by r. We let $\xi\in X_t$ if and only if $\mathrm{reg}((s^G_1)^\frown r)^\frown\langle\xi\rangle$ is coherent with respect to G_β and $\mathrm{reg}((s^H_0)^\frown r)^\frown\langle\xi\rangle$ is coherent with respect to H_β and $\xi>\Xi$.

(3) Suppose that **Case C** holds. We let $\xi\in X_t$ if and only if $t^\frown\langle\xi\rangle$ is coherent with respect to both G_β and H_β and $\xi>\Xi$.

Subclaim 4.16.1.1. *Let $\tau \leq o^{\mathcal{U}}(\beta)$. The set X_t is a member of $\mathcal{U}_{\tau,\sigma}(t)$.*

Proof. Since $\Xi < \beta$, the requirement $\xi > \Xi$ can easily be satisfied, so we ignore it.

Suppose first that **Case A** holds, witnessed by r. By Lemma 3.4 (3), the collection Y_t of all ordinals ξ such that $t^\frown\langle\xi\rangle = \mathrm{reg}((s_0^G)^\frown r)^\frown\langle\xi\rangle$ is coherent with respect to G_β is a member of $\mathcal{U}_{\beta,\tau}(t)$. By the same argument, the collection Z_t of all ordinals ξ such that $\sigma^\tau_\beta(t)^\frown\langle\xi\rangle = \mathrm{reg}((s_1^H)^\frown r)^\frown\langle\xi\rangle$ is coherent with respect to H_β is a member of $\mathcal{V}_{\beta,\tau}(\sigma^\tau_\beta(t))$, which is equal to $\mathcal{U}_{\beta,\tau}(t)$ (by Claim 4.15.1). Thus, $X_t = Y_t \cap Z_t \in \mathcal{U}_{\beta,\tau}(t)$.

Case B is entirely symmetric to **Case A**.

Suppose that **Case C** holds. Again, the collection Y_t of all ordinals ξ such that $t^\frown\langle\xi\rangle$ is coherent with respect to G_β is in $\mathcal{U}_{\beta,\tau}(t)$ and the collection Z_t of all ordinals ξ such that $t^\frown\langle\xi\rangle$ is coherent with respect to H_β is in

$$\mathcal{V}_{\beta,\tau}(t) = \mathcal{U}_{\beta,\tau}(\sigma^\tau_\beta(t)) = \mathcal{U}_{\beta,\tau}(t).$$

Here, the last equality follows since $\sigma^\tau_\beta(t)$ is equivalent to t. Therefore, $X_t := Y_y \cap Z_t \in \mathcal{U}_{\beta,\tau}(t)$. □

Now we are ready to define the pure extension $\langle t, T^*\rangle$ lying in D_β.

(1) Suppose that **Case A** holds, witnessed by r. We let $s \in T^*$ if and only if $s \in T$, $\mathrm{reg}((s_0^G)^\frown r)^\frown s$ is coherent with respect to G_β and $\mathrm{reg}((s_1^H)^\frown r)^\frown s$ is coherent with respect to H_β and $\min(s) > \Xi$.
(2) Suppose that **Case B** holds, witnessed by r. We let $s \in T^*$ if and only if $s \in T$, $\mathrm{reg}((s_1^G)^\frown r)^\frown s$ is coherent with respect to G_β and $\mathrm{reg}((s_0^H)^\frown r)^\frown s$ is coherent with respect to H_β and $\min(s) > \Xi$.
(3) Suppose that **Case C** holds. We let $s \in T^*$ if and only if $s \in T$, $t^\frown s$ is coherent with respect to both G_β and H_β and $\min(s) > \Xi$.

It is clear that, if $\langle t, T^*\rangle \in \mathbb{Q}_\beta$, it is an element of D_β and a pure extension of $\langle t, T\rangle$. Therefore, we are done after showing:

Subclaim 4.16.1.2. $\langle t, T^*\rangle \in \mathbb{Q}_\beta$.

Proof. The only substantial claim is showing that for all $s \in T^*$, $\mathrm{Suc}_{T^*}(s) \in \mathcal{U}_{\beta,\tau}(t^\frown s)$ for every $\tau < o^{\mathcal{U}}(\beta)$. Crucially, the case we are in for t is the same case we are in for $t^\frown s$: If we are in **Case A** or B, witnessed by r, $r^\frown s$ witnesses that we are in **Case A** or **Case B** respectively. If we are in **Case C**, we are still in **Case C** for $r^\frown s$, since $\min(r) > \Xi$.

So suppose we are in **Case A**. Then $\mathrm{Suc}_{T^*}(s)$ consists of all $\xi \in \mathrm{Suc}_{T^*}(s)$ such that $\mathrm{reg}((s_0^G)^\frown r)^\frown r^\frown\langle\xi\rangle$ is coherent with respect to G_β and the sequence $\mathrm{reg}((s_1^H)^\frown r)^\frown s^\frown\langle\xi\rangle$ is coherent with respect to H_β. Since $\mathrm{reg}(x_0)^\frown x_1^\frown\langle\zeta\rangle$ is coherent with respect to a filter if and only if this is true for $\mathrm{reg}(x_0^\frown x_1)^\frown\langle\zeta\rangle$, we must have $\mathrm{Suc}_{T^*}(s) = \mathrm{Suc}_T(s) \cap X_{t^\frown s}$, which is as required.

The same argument applies equally in **Case B** and **Case C**. □Subclaim 4.16.1.2

This finishes the proof of the claim. □Claim 4.16.1

Now define the putative isomorphism π_β as in Clause ($\beth$); namely,

$$\pi_\beta(\langle t, T\rangle) = \langle \sigma_\beta^{o^{\mathcal{U}}(\beta)}(t), T\rangle \text{ for } \langle t, T\rangle \in D_\beta.$$

Claim 4.16.2. *π_β is a map from D_β into $(\Gamma^{-1}_{\vec{\beta}})^*(D_\beta)^{G_\beta}$.*

Proof. First, notice that $(\Gamma^{-1}_{\vec{\beta}})^*(D_\beta)^{G_\beta} = \dot{D}_\beta^{H_\beta}$, i.e. $(\Gamma^{-1}_{\vec{\beta}})^*(D_\beta)^{G_\beta}$ is defined the same way as D_β, except replacing G_β by H_β and thus replacing H_β by G_β (because Γ is self-inverse modulo equivalence).

So suppose that $\langle t, T\rangle \in D_\beta$. We check the three cases.

(1) Suppose that **Case A** holds for t, witnessed by r, i.e. $t = \mathrm{reg}((s_0^G)^\frown r)$ and $\sigma_\beta^{o(\beta)}(t) = \mathrm{reg}((s_1^H)^\frown r)$. Let $s := \sigma_\beta^{o(\beta)}(t)$. It follows that, in $V[H_\beta]$, **Case B** holds for s, witnessed by r as well, since the roles of G_β and H_β are switched in $V[H_\beta]$. Then, because **Case A** holds for t in $V[G_\beta]$, whenever $u \in T$, $\mathrm{reg}((s_0^G)^\frown r)^\frown s$ is coherent with respect to G_β and $\mathrm{reg}((s_1^H)^\frown r)^\frown s$ is coherent with respect to H_β. Therefore, $\langle s, T\rangle \in \dot{D}_\beta^{H_\beta}$ by the definition.

(2) Suppose that **Case B** holds for t, witnessed by r, i.e. $t = \mathrm{reg}((s_1^G)^\frown r)$ and $\sigma_\beta^{o(\beta)}(t) = \mathrm{reg}((s_0^H)^\frown r)$. Let $s := \sigma_\beta^{o(\beta)}(t)$. It follows that, in $V[H_\beta]$, **Case A** holds for s, witnessed by r as well, since the roles of G_β and H_β are switched in $V[H_\beta]$. Then, because **Case B** holds for t in $V[G_\beta]$, whenever $u \in T$, $\mathrm{reg}((s_1^G)^\frown r)^\frown s$ is coherent with respect to G_β and $\mathrm{reg}((s_0^H)^\frown r)^\frown s$ is coherent with respect to H_β. Therefore, $\langle s, T\rangle \in \dot{D}_\beta^{H_\beta}$ by the definition.

(3) Suppose that **Case C** holds for t. It follows that, in $V[H_\beta]$, **Case C** holds for t as well, since a witness e.g. for **Case A** in $V[H_\beta]$ would yield a witness for **Case B** in $V[G_\beta]$ and vice versa. Thus, whenever $s \in T$, $t^\frown s$ is coherent with respect to both G_β and H_β and $\min(s) > \Xi$, showing that $\langle t, T\rangle \in \dot{D}_\beta^{G_\beta}$. □

Claim 4.16.3. *$\pi_\beta \circ \Gamma^*(\dot{\pi}_\beta)^{G_\beta} = \mathrm{id}_{\Gamma^*(\dot{D}_\beta)^{G_\beta}}$.*

Proof. Since G_β was arbitrary, using the previous observation, it is sufficient (and equivalent) to show that $\dot{\pi}_\beta^{H_\beta} \circ \pi_\beta = \mathrm{id}_{D_\beta}$. So let $\langle t, T\rangle \in D_\beta$.

(1) Suppose that **Case A** holds for t, witnessed by r, i.e. $t = \mathrm{reg}((s_0^G)^\frown r)$ and $\sigma_\beta^{o(\beta)}(t) = \mathrm{reg}((s_1^H)^\frown r)$. Let $s := \sigma_\beta^{o(\beta)}(t)$. It follows that, in $V[H_\beta]$, **Case B** holds for s, witnessed by r as well, since the roles of G_β and H_β are switched in $V[H_\beta]$. Since $s = \mathrm{reg}((s_1^H)^\frown r)$, $(\dot{\sigma}_\beta^{o(\beta)})^{H_\beta}(s) = \mathrm{reg}((s_0^G)^\frown r) = t$. Thus, $\dot{\pi}_\beta^{H_\beta}(\pi_\beta(\langle t, T\rangle)) = \langle t, T\rangle$ by the definition.

(2) Suppose that **Case B** holds for t, witnessed by r, i.e. $t = \mathrm{reg}((s_1^G)^\frown r)$ and $\sigma_\beta^{o(\beta)}(t) = \mathrm{reg}((s_0^H)^\frown r)$. Let $s := \sigma_\beta^{o(\beta)}(t)$. It follows that,

in $V[H_\beta]$, **Case A** holds for s, witnessed by r as well, since the roles of G_β and H_β are switched in $V[H_\beta]$. Since $s = \mathrm{reg}((s_0^H)^\frown r)$, $(\dot\sigma_\beta^{o(\beta)})^{H_\beta}(s) = \mathrm{reg}((s_1^G)^\frown r) = t$. Thus, $\dot\pi_\beta^{H_\beta}(\pi_\beta(\langle t, T\rangle)) = \langle t, T\rangle$ by the definition.

(3) Suppose that **Case C** holds for t. It follows that, in $V[H_\beta]$, **Case C** holds for t as well. Therefore,

$$\dot\pi_\beta^{H_\beta}(\pi_\beta(\langle t, T\rangle)) = \dot\pi_\beta^{H_\beta}(\langle t, T\rangle) = \langle t, T\rangle$$

as desired. □

It is clear from the definition that π_β preserves the pure extension ordering. Let us show that π_β preserves the normal ordering:

Claim 4.16.4. *Let $\langle s, S\rangle, \langle t, T\rangle \in D_\beta$ and suppose that $\langle s, S\rangle \leq_{D_\beta} \langle t, T\rangle$. Then $\pi_\beta(\langle s, S\rangle) \leq_{\dot D_\beta^{H_\beta}} \pi_\beta(\langle t, T\rangle)$*

Proof. By the definition of $\leq_{\mathbb{Q}_\beta}$ there exists $\eta \in T$ such that $\langle s, S\rangle \leq_{\mathbb{Q}_\beta^*} \langle t^\frown\eta, T_\eta\rangle$. By the definition of D_β, s is minimal and so we must have $s = \mathrm{reg}(t^\frown\eta)$.

So suppose that **Case A** holds for t, witnessed by r. By the definition of D_β it follows that **Case A** also holds for $s = \mathrm{reg}(t^\frown\eta)$, witnessed by $r^\frown\eta$. Ergo we have

$$\sigma_\beta^{o(\beta)}(s) = \mathrm{reg}((s_1^H)^\frown r^\frown\eta) = \mathrm{reg}(\mathrm{reg}((s_1^H)^\frown r)^\frown\eta) = \mathrm{reg}(\sigma_\beta^{o^{\mathcal{U}}(\beta)}(t)^\frown\eta)$$

which shows that

$$\pi_\beta(\langle s, S\rangle) = \langle \sigma_\beta^{o(\beta)}(s), S\rangle \leq_{\dot D_\beta^{H_\beta}} \langle \sigma_\beta^{o(\beta)}(t), T\rangle = \pi_\beta(\langle t, T\rangle)$$

as desired.

The same argument works for **Case B** and **Case C** holding for t as well. In the latter case we use that, because $\min(s) > \xi$, if **Case C** holds for t, **Case C** also holds for $t^\frown\eta$. □

Now suppose that $o^{\mathcal{U}}(\beta) < \beta^+$. We are in a position to conclude the proof of the induction:

Claim 4.16.5. *For $o^{\mathcal{U}}(\beta) < \beta^+$, $(\dot\pi_\gamma, \dot D_\gamma)_{\gamma<\beta+1}$ witnesses Clauses (ℵ)–(ℶ).*

Proof of claim. Clause (ℶ) is clear, by construction. Let us check Clause (ℵ); namely, $(\dot\pi_\gamma, \dot D_\gamma)_{\gamma<\beta+1}$ is an automorphism sequence: First, observe that the previous construction works irrespective of the chosen generic filter G_β, ergo the trivial condition of $\mathbb{P}_\beta$ forces all the claims established so far. In particular, invoking Claim 4.16.1 $\dot D_\beta$ is forced to be an open $\leq^*_{\mathbb{Q}_\beta}$-dense subset of $\mathbb{Q}_\beta$. Second, Claim 4.16.4 and the comments preceding it showed that $\dot\pi_\beta$ is forced to be an isomorphism from $\dot D_\beta$ onto $\Gamma^*_{\vec\pi\restriction\beta}(\dot D_\beta)$ such that $\dot\pi_\beta \circ \Gamma^*_{\vec\pi\restriction\beta}(\dot\pi_\beta) = \mathrm{id}_{\dot D_\beta}$. □

Now suppose that $o^{\mathcal{U}}(\beta) = \beta^+$. As before, we have that π_β is an isomorphism of D_β and $(\Gamma^{-1}_{\vec{\pi}})^*(\dot{D}_\beta)^{G_\beta}$. Let us call the dense set E_β and the automorphism τ_β instead so that we can use D_β and π_β again. Suppose first that $\beta \notin A$. We let π_β be the automorphism of $\mathbb{R}_\beta = \mathbb{Q}_\beta^{\beta^+} * \dot{\mathbb{C}}_\beta$ defined by $\pi_\beta(\langle t, T\rangle, \dot{r}) = (\langle t, T\rangle, \dot{r})$. Now suppose that $\beta \in A$. For now, work in V. Recall that $q_i(\beta) = (\langle \dot{t}^i_\beta, \dot{T}^i_\beta\rangle, \dot{r}^i_\beta)$.

Recall that $\mathbb{P}_\beta$ has the β^+-cover property by Corollary 2.30 and forces that $\dot{\mathbb{Q}}_\beta^{\beta^+}$ is β^+-cc by Lemma 3.10. In V, fix a well-ordering $\prec_0$ on $[\beta^+]^\beta$ and fix a well-ordering $\prec_1$ on ${}^{\beta^+}\beta^+$. In $V[G_\beta]$, let $X_\beta \in V$ be the $\prec_0$-least set such that for some $p \in G_\beta$,

$$(p, \mathbb{1}_{\dot{\mathbb{Q}}_\beta^{\beta^+}}) \Vdash_{\mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta^{\beta^+}} \mathrm{supp}(\dot{r}^0_\beta) \cup \mathrm{supp}(\dot{r}^1_\beta) \subseteq \check{X}_\beta$$

Also let Y_β be the $\prec_0$-least set such that for some $q \in H_\beta$,

$$(q, \mathbb{1}_{\dot{\mathbb{Q}}_\beta^{\beta^+}}) \Vdash_{\mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta^{\beta^+}} \mathrm{supp}(\dot{r}^0_\beta) \cup \mathrm{supp}(\dot{r}^1_\beta) \subseteq \check{Y}_\beta$$

In $V[G_\beta]$, let ι_β be the $\prec_1$-least self-inverse bijection from β^+ to β^+ such that $\iota_\beta[X_\beta \cup Y_\beta]$ is disjoint from $X_\beta \cup Y_\beta$. Clearly, $\iota_\beta^2 = \mathrm{id}_{X_\beta}$. In $V[G_\beta][K_\beta]$, where K_β is $\mathbb{Q}_\beta^{\beta^+}$-generic over $V[G_\beta]$, let σ_{ι_β} be the automorphism of $\dot{\mathbb{R}}_\beta^{K_\beta}$ such that, for every $r \in \dot{\mathbb{R}}_\beta^{K_\beta}$, $\mathrm{supp}(\sigma_{\iota_\beta}(r)) = \iota_\beta[\mathrm{supp}(r)]$ and $\sigma_{\iota_\beta}(r)(\iota_\beta(\gamma)) = r(\gamma)$. Let $\dot{\sigma}_{\iota_\beta}$ be a name for σ_{ι_β}. Let D_β consist of all $(q, r) \in \mathbb{Q}_\beta^{\beta^+} * \dot{\mathbb{R}}_\beta$ such that $q \in E_\beta$. Since E_β is $\leq^*_{\mathbb{Q}_\beta^{\beta^+}}$-dense by Claim 4.16.1, it follows that D_β is $\leq^*_\beta$-dense. For $(q, \dot{r}) \in D_\beta$, let $\pi_\beta(q, \dot{r}) := (\tau_\beta(q), \tau^*_\beta(\dot{\sigma}_{\iota_\beta}(\dot{r}))$.

Claim 4.16.6. *For $o^{\mathcal{U}}(\beta) \geq \beta^+$, $(\dot{\pi}_\gamma, \dot{D}_\gamma)_{\gamma<\beta+1}$ witnesses Clauses (א)–(ב).*

Proof. As in the previous proof, since G_β was arbitrary, it suffices to show that the desired statements hold in $V[G_\beta]$.

Subclaim 4.16.6.1. *π_β is an isomorphism of D_β and $(\Gamma^{-1}_{\vec{\pi}})^*(\dot{D}_\beta)^{G_\beta}$.*

Proof. By the definition, $(\Gamma^{-1}_{\vec{\pi}})^*(\dot{D}_\beta)^{G_\beta} = \dot{D}_\beta^{H_\beta}$ consists of all $(q, \dot{c})$ such that $q \in \dot{E}_\beta^{H_\beta}$ and $q \Vdash \dot{c} \in (\dot{\mathbb{C}}_\beta^{H_\beta})$ (here we regard $\dot{\mathbb{C}}_\beta$ as a $\mathbb{P}_\beta$-name for a $\dot{\mathbb{Q}}_\beta^{\beta^+}$-name), i.e. $q \in \dot{E}_\beta^{H_\beta}$ and q forces that $\dot{c}$ is a $< \beta^+$-supported function on β^+ that maps into the collection of closed bounded subsets of β consisting of V-measurable cardinals. As we have seen before, if $(q, \dot{c}) \in D_\beta$, then $\tau_\beta(q) \in \dot{E}_\beta^{H_\beta}$. Since $(q, \dot{c}) \in \mathbb{Q}_\beta^{\beta^+}$, q forces that $\dot{c}$ and thus $\dot{\sigma}_{\iota_\beta}(\dot{c})$ is a $< \beta^+$-supported function from β^+ into the collection of closed bounded subsets of β consisting of V-measurable cardinals. Therefore, $\tau_\beta(q)$ forces the same of $\tau^*_\beta(\dot{\sigma}_{\iota_\beta}(\dot{r}))$, showing that $(\tau_\beta(q), \tau^*_\beta(\dot{\sigma}_{\iota_\beta}(\dot{c}))) \in \dot{E}_\beta^{H_\beta}$ as desired.

A similar argument shows that π_β preserves the pure extension ordering on D_β and $(\Gamma^{-1}_{\vec{\pi}})^*(\dot{D}_\beta)^{G_\beta}$. The more substantial claim is showing that the

regular ordering is preserved. Recall that we defined $(q, \dot{c}) \leq_{\mathbb{R}_\beta} (q, \dot{c}')$ if $\pi(q, \dot{r}) \leq_{\mathrm{RO}^+(\mathbb{C}_\beta)} \pi(q', \dot{r}')$. By Lemma 3.23, we have

$$\pi((q, \dot{c})) = \bigwedge \{b \in \mathrm{RO}^+(\mathbb{C}_\beta) \mid q \Vdash \dot{c} \leq_{\mathrm{RO}^+(\mathbb{C}_\beta)} \check{b}\}$$

To this end, let γ_{ι_β} be the map on $\mathbb{C}_\beta$ defined as before: For $r \in \mathbb{C}_\beta$, $\gamma_{\iota_\beta}(r)$ is a function with support $\iota_\beta[\mathrm{supp}(r)]$ such that $\gamma_{\iota_\beta}(r)(\iota_\beta(\gamma)) = r(\gamma)$. We lift γ_{ι_β} to a function on $\mathrm{RO}^+(\mathbb{C}_\beta)$ (which we denote by the same symbol for simplicity) as follows: For $b \in \mathrm{RO}^+(\mathbb{C}_\beta)$, let $b = \bigwedge A$, where $A \subseteq \mathbb{C}_\beta$ is an antichain. We let $\gamma_{\iota_\beta}(b) := \bigwedge \gamma_{\iota_\beta}[A]$. It is easy to see that γ_{ι_β} is an automorphism of $\mathrm{RO}^+(\mathbb{C}_\beta)$.

Subsubclaim 4.16.6.1.1. *For every $(q, \dot{c}) \in D_\beta$,*

$$\dot{\pi}^{H_\beta}(\pi_\beta(q, \dot{c})) = \gamma_{\iota_\beta}(\pi(q, \dot{c})).$$

Proof. Note that $\pi_\beta(q, \dot{c}) = (\tau_\beta(q), \tau^*_\beta(\dot{\sigma}_\beta(\dot{c})))$. Let $b \in \mathrm{RO}^+(\mathbb{C}_\beta)$ and suppose that

$$\tau_\beta(q) \Vdash \tau^*_\beta(\dot{\sigma}_{\iota_\beta}(\dot{c})) \leq_{\mathrm{RO}^+(\dot{\mathbb{C}}_\beta)} \check{b}$$

In particular, we have

$$q \Vdash \dot{\sigma}_{\iota_\beta}(\dot{c}) \leq_{\mathrm{RO}^+(\dot{\mathbb{C}}_\beta)} \check{b}$$

since the definition of the ordering on $\mathrm{RO}^+(\dot{\mathbb{C}}_\beta)$ is absolute (it is defined simply by reverse inclusion on the order and pointwise end-extension). We claim that this implies that $q \Vdash \dot{c} \leq_{\mathrm{RO}^+(\dot{\mathbb{C}}_\beta)} \check{\gamma}_{\iota_\beta}(\check{b})$. As in the proof of Lemma 3.23, it suffices to show that $(q, \dot{c})$ forces that $\check{\gamma}_{\iota_\beta}(\check{b})$ belongs to the induced generic filter on $\mathrm{RO}^+(\dot{\mathbb{C}}_\beta)$.

To this end, let $K * L$ be $\mathbb{Q}_\beta^{\beta^+} * \dot{\mathbb{C}}_\beta$-generic containing q. It follows that $K * \sigma_{\iota_\beta}[L]$ is $\mathbb{Q}_\beta^{\beta^+} * \dot{\mathbb{C}}_\beta$-generic containing $(q, \dot{\sigma}_{\iota_\beta}(\dot{c}))$. In particular in the model $V[K * \sigma_{\iota_\beta}[L]]$, b is in $\dot{G}^{K * \sigma_{\iota_\beta}[L]}_{\mathrm{RO}^+(\mathbb{C}_\beta)}$. (Recall that $\dot{G}^{K * \sigma_{\iota_\beta}[L]}_{\mathrm{RO}^+(\mathbb{C}_\beta)}$ is simply the upward closure of $\sigma_{\iota_\beta}[L]$, since $\sigma_{\iota_\beta}[L] \subseteq \mathbb{C}_\beta$.) Thus, there must be $x \in L$ such that $\sigma_{\iota_\beta}(x) \leq b$. Now, by the definition of σ_{ι_β} and γ_{ι_β}, we have $\sigma_{\iota_\beta}(x) = \gamma_{\iota_\beta}(x)$ for all $x \in \dot{\mathbb{C}}^K_\beta$. Since γ_{ι_β} is lifted to $\mathrm{RO}^+(\mathbb{C}_\beta)$ and it is a self-inverse automorphism, we have $x \leq \gamma_{\iota_\beta}(b)$ and so $\gamma_{\iota_\beta}(b) \in \dot{G}^{K*L}_{\mathrm{RO}^+(\mathbb{C}_\beta)}$. As $K * L$ was arbitrary, we are done.

Now by symmetry, it follows that

$$q \Vdash_{\mathbb{Q}_\beta^{\beta^+}} \dot{\sigma}_{\iota_\beta}(\dot{c}) \leq_{\mathrm{RO}^+(\mathbb{C}_\beta)} \check{b} \text{ if and only if } q \Vdash_{\mathbb{Q}_\beta^{\beta^+}} \dot{c} \leq_{\mathrm{RO}^+(\mathbb{C}_\beta)} \check{\gamma}_{\iota_\beta}(\check{b})$$

proving the desired statement. $\square_{\text{Subsubclaim 4.16.6.1.1}}$

Now we can finish the proof: Let $(q', \dot{c}') \leq_{\dot{\mathbb{R}}_\beta} (q, \dot{c})$, i.e. $\pi(q', \dot{c}') \leq_{\mathrm{RO}^+(\mathbb{C}_\beta)} \pi(q, \dot{c})$. Therefore, we have

$$\dot{\pi}^{H_\beta}(\pi_\beta(q', \dot{c}')) = \gamma_{\iota_\beta}(\pi(q', \dot{c}')) \leq_{\mathrm{RO}^+(\mathbb{C}_\beta)} \gamma_{\iota_\beta}(\pi(q, \dot{c})) = \dot{\pi}^{H_\beta}(\pi_\beta((q, \dot{c})))$$

showing that $\pi_\beta(q', \dot{c}') \leq \pi_\beta(q, \dot{c})$. $\square_{\text{Subclaim 4.16.6.1}}$

The above proof also shows that π_β has the form stipulated in ($\beth$).

Finally, we must show the following:

Subclaim 4.16.6.2. $\pi_\beta \circ \Gamma^*_{\vec{\pi}}(\dot{\pi}_\beta)^{G_\beta} = \mathrm{id}_{\Gamma^*_{\vec{\pi}}(\dot{D}_\beta)^{G_\beta}}$.

Proof. As in the proof of Claim 4.16.3, it suffices to show instead that $\dot{\pi}_\beta^{H_\beta} \circ \pi_\beta = \mathrm{id}_{D_\beta}$. By Claim 4.16.3, we have that $\dot{\tau}_\beta^{H_\beta} \circ \tau_\beta = \mathrm{id}_{E_\beta}$.

Second, we show that $\iota_\beta = \dot{\iota}_\beta^{H_\beta}$: Recalling the definition of ι_β, we chose the $\prec_0$-least X_β and Y_β that were forced by conditions inside of G_β and H_β respectively to cover $\mathrm{supp}(\dot{r}^i_\beta)$ for $i < 2$. It follows that, in $V[H_\beta]$, since the roles of G_β and H_β are reversed, $\dot{X}_\beta^{H_\beta} = Y_\beta$ and $\dot{Y}_\beta^{H_\beta} = X_\beta$. In particular, their union is the same. Thus, since ι_β was chosen in relation to $X_\beta \cup Y_\beta$ and the ground-model well-ordering, we have $\dot{\iota}_\beta^{H_\beta} = \iota_\beta$. Now it follows as in the proof of Lemma 4.3 that $\dot{\pi}_\beta^{H_\beta} \circ \pi_\beta = \mathrm{id}_{D_\beta}$. □

This shows that $(\dot{\pi}_\gamma, \dot{D}_\gamma)_{\gamma<\beta+1}$ witnesses clauses ($\aleph$) and ($\beth$) in the case $o^{\mathcal{U}}(\beta) \geq \beta^+$, which was our goal. □Claim 4.16.6

Thereby, the induction has been accomplished and we obtain an automorphism sequence $(\dot{\pi}_\alpha, \dot{D}_\alpha)_{\alpha<\kappa+1}$ witnessing ($\aleph$) and ($\beth$).

We are now in a position to prove that the conclusion of almost homogeneity modulo equivalence holds for $\mathbb{P}_{\kappa+1}$ and $q_0, q_1 \in \mathbb{P}_{\kappa+1}$. Let $w_0 \leq^* q_0$ be an element of $\mathcal{D}(\vec{D})$. Our goal is to show that $\Gamma_{\vec{\pi}}(w_0)$ is $\leq^*$-compatible with q_1. To this end, it suffices to show the following for every $\alpha \in A$:[20]

(1) If $o(\alpha) < \alpha^+$, then $\Gamma_{\vec{\pi}}(w_0)(\alpha)$ is forced by $\Gamma_{\vec{\pi}\restriction\alpha}(w_0 \restriction \alpha)$ to have the same stem as $q_1(\alpha)$.
(2) If $o^{\mathcal{U}}(\alpha) = \alpha^+$, then if $\Gamma_{\vec{\pi}}(w_0)(\alpha) = (\langle \dot{s}^0_\alpha, \dot{S}^0_\alpha\rangle, \dot{u}^0_\alpha)$ and $q_1(\alpha) = (\langle \dot{t}^1_\alpha, \dot{T}^1_\alpha\rangle, \dot{r}_1)$, it is forced by $\Gamma_{\vec{\pi}\restriction\alpha}(w_0 \restriction \alpha)$ that $\dot{s}^0_\alpha = \dot{t}^1_\alpha$ and that $\langle \dot{t}^1_\alpha, \dot{T}^1_\alpha\rangle$ forces that $\dot{u}^0_\alpha$ and $\dot{r}^1_\alpha$ are compatible.

So let H_α be any $\mathbb{P}_\alpha$-generic filter containing $\Gamma_{\vec{\pi}\restriction\alpha}(w_0 \restriction \alpha)$ and let $G_\alpha := \Gamma^{-1}_{\vec{\pi}\restriction\alpha}[H_\alpha]$. By the construction,

$$\Gamma_{\vec{\pi}}(w_0)(\alpha)^{H_\alpha} = \pi_\alpha(\Gamma^*_{\vec{\pi}\restriction\alpha}(w_0(\alpha))^{H_\alpha}) = \pi_\alpha(w_0(\alpha)^{G_\alpha})$$

Suppose that $o(\alpha) < \alpha^+$. By construction, the stems of $w_0(\alpha)^{G_\alpha}$ and $(q_0(\alpha))^{G_\alpha} = (h_0(\alpha))^{G_\alpha}$ are equal. Ergo, by the inductive hypothesis placed on π_α, the stem of $\pi_\alpha(q_0(\alpha)^{G_\alpha})$ is equal to $(h_1(\alpha))^{H_\alpha}$. Note that $h_1(\alpha)$ was a name for the stem of $q_1(\alpha)$. Since H_α was arbitrary, it follows that $\Gamma_{\vec{\pi}\restriction\alpha}(w_0 \restriction \alpha)$ forces that the stems of $\Gamma_{\vec{\pi}}(w_0)(\alpha)$ and $q_1(\alpha)$ are equal, just as desired.

Suppose that $o^{\mathcal{U}}(\alpha) = \alpha^+$. Just like before, $(\dot{s}^0_\alpha)^{H_\alpha} = \dot{s}_1^{H_\alpha}$. Furthermore, by the inductive hypothesis, it is forced by $\mathbb{Q}^{\alpha^+}_\alpha$ over $V[H_\alpha]$ that $\mathrm{supp}(\dot{u}^0_\alpha) =$

[20]Recall that $\mathrm{supp}(q_0) = \mathrm{supp}(q_1)$ and that A was their common value.

$\iota_\beta[\mathrm{supp}(\dot r^0_\alpha)]$ and that this is disjoint from $\mathrm{supp}(\dot r^1_\alpha)$. Therefore, it is forced by $\mathbb{Q}^{\alpha^+}_\alpha$ that $\dot u^0_\alpha$ and $\dot r^1_\alpha$ are compatible.

This finishes the proof. $\square_{\text{Theorem 4.10}}$

5. Preserving HOD-supercompactness

In this section we conclude the proof of our main theorem. To this end, assume the hypotheses from Setup 3, and let $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ be the forcing constructed in Section 3. The main goal in this section is showing that after forcing with $\mathbb{P}$, κ is HOD-supercompact.

Lemma 5.1. *$\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ preserves cardinals.*

Proof. This is immediate from Lemmas 2.32 and 3.26. $\square$

Theorem 5.2. *Assume that κ is a supercompact cardinal and* GCH *holds. Let $G * g \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$ be generic over V. Then κ remains supercompact in $V[G * g]$. Furthermore, if, in addition, $V = g\mathrm{HOD}$ holds in V, κ is* HOD*-supercompact in $V[G * g]$.*

Proof. Let $\lambda > \kappa$ be such that for every fine normal measure $\mathcal{V}$ on $\mathcal{P}_\kappa(\lambda)$,

$$j_{\mathcal{V}}(\mathcal{U}) \restriction \kappa + 1 = \mathcal{U}.$$

Let $\mathcal{V}$ be a fine normal measure on $\mathcal{P}_\kappa(\lambda)$ such that $j_{\mathcal{V}}(\ell)(\kappa) = \lambda$.

To simplify forthcoming notations, let us denote $j := j_{\mathcal{V}}$ and $M := M_{\mathcal{V}}$.

Since $j(\mathcal{U}) \restriction \kappa + 1 = \mathcal{U}$, we obtain that

$$j(\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa) \restriction \kappa + 1 = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa.$$

We can thus naturally factor the iteration as

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{G}}_\kappa * j(\mathbb{P}_\kappa)/\dot G * \dot g.$$

Let us exhibit a normal, fine, κ-complete ultrafilter on $\mathcal{P}_\kappa^{V[G*g]}(\lambda)$ in the extension $V[G * g]$. We begin constructing a suitable master condition.

Claim 5.2.1. *There exists a $j(\mathbb{P}_\kappa) \restriction \kappa+1$-name $\dot m \in M$ such that $(p, \dot q, \dot m) \leq j(p, \dot q)$ for all $(p, \dot q) \in G * g$.*

Proof. The master condition $\dot m$ we will construct will have the form $(\dot m_0, \dot m_1)$, where it is forced that $\dot m_0 \in j(\mathbb{P}_\kappa) \setminus \kappa+1$, and $\mathbb{1}^\frown \dot m_0$ forces that $\dot m_1 \in j(\dot{\mathbb{Q}}_\kappa)$.

Construction of m_0: Recall that for every active forcing coordinate $\alpha \in \mathrm{dom}_1(\mathcal{U}) \cup \{\kappa\}$, the poset $\mathbb{Q}_\alpha$ has a $\leq^*$-dense subset $\mathbb{D}_\alpha$ which is $< \alpha$-directed-closed with respect to the pure extension order $\leq^*_{\mathbb{Q}_\alpha}$ (see Lemma 3.21).

Let

$$D = \{p \in \mathbb{P}_\kappa \mid \forall \alpha \in \mathrm{supp}(p), p \restriction \alpha \Vdash p(\alpha) \in \dot{\mathbb{D}}_\alpha\}.$$

Clearly, $D \subseteq \mathbb{P}_\kappa$ is $\leq^*$-dense. Over $M[G * g]$, denote

$$j[G] \setminus (\kappa + 1) := \{j(p) \setminus (\kappa + 1) \mid p \in G\}.$$

$$j[G \cap D] \setminus (\kappa + 1) := \{j(p) \setminus (\kappa + 1) \mid p \in G \cap D\}.$$

By supercompactness, $j[G \cap D] \setminus (\kappa+1) \in M[G*g]$. Let us argue that this is a $\leq^*$-directed set of conditions. In fact, $j[G] \setminus (\kappa+1)$ is a $\leq^*$-directed set of conditions: Indeed, given $p_0, p_1 \in G$, let $p^* \in G$ be a common extension. By the definition of extension, for all but finitely many values of $\alpha \in \operatorname{supp}(p_0) \cup \operatorname{supp}(p_1)$,

$$p \restriction \alpha \Vdash p(\alpha) \leq^* p_0(\alpha), p_1(\alpha).$$

In particular, $j(p) \setminus (\kappa+1) \leq^* j(p_0) \setminus (\kappa+1), j(p_1) \setminus (\kappa+1)$ as needed.

In $V[G*g]$, $|j[G \cap D] \setminus (\kappa+1)| = \lambda$. Since $j(\ell)(\kappa) = \lambda$, the forcing $j(\mathbb{P}) \restriction (\kappa, \lambda]$ does not contain active stages. In particular, each poset in $j(\langle \mathbb{D}_\alpha : \alpha < \kappa \rangle) \setminus (\kappa+1)$ has a direct extension order which is λ^+-directed closed. Since $V[G*g] \vDash {}^{\lambda}M[G*g] \subseteq M[G*g]$, $\langle j(D) \setminus (\kappa+1), \leq^* \rangle$ is λ^+-directed closed in $V[G*g]$. It thus follows that there exists a condition $m_0 \in j(\mathbb{P}) \setminus (\kappa+1)$ such that, for every $p \in G \cap D$, $m_0 \leq j(p) \setminus (\kappa+1)$. By density of D, we deduce that for every $p \in G$, $m_0 \leq j(p) \setminus (\kappa+1)$.

Construction of $\dot{m}_1$: Work in $V[G]$. The generic $g \subseteq \dot{\mathbb{Q}}_\kappa^G$ induces a sequence $\langle c_\alpha^g : \alpha < \kappa^+ \rangle$ of clubs on κ. We construct $\dot{m}_1$ by defining a $j(\mathbb{P}_\kappa)$-name for a condition $\dot{m}_1^*$ forced in $\dot{\mathbb{Q}}_{j(\kappa)}$ which is projected onto $\dot{m}_1$. Denote $\mathbb{1}$ the trivial condition of $\mathbb{Q}_{j(\kappa)}^{j(\kappa)^+}$ and let $\dot{\delta}$ be the $\mathbb{Q}_{j(\kappa)}^{j(\kappa)^+}$-name for the least M-measurable cardinal above κ in the generic club subset of $j(\kappa)$ that is induced by the generic for $\mathbb{Q}_{j(\kappa)}^{j(\kappa)^+}$. Let $m_1^* = (\mathbb{1}, \dot{x})$ where $\dot{x}$ is forced to be a function with domain $j[\kappa^+] \in ([j(\kappa)^+]^{<j(\kappa)})^M$, and for every $\alpha < \kappa^+$,

$$\dot{x}(j(\alpha)) = c_\alpha^g \cup \{\check{\kappa}\} \cup \{\dot{\delta}\}.^{21}$$

This concludes the definition of $\dot{m} = (\dot{m}_0, \dot{m}_1)$.

Verification: It now remains to verify that for every $(p, \dot{q}) \in G*g$,

$$(p, \dot{q}, \dot{m}) \leq j(p, \dot{q}).$$

Fix $(p, \dot{q}) \in G*g$. Since $\mathbb{P}_\kappa$ has a nonstationary support, $\kappa \notin \operatorname{supp}(j(p))$, and thus, by the choice of $\dot{m}_0$, we have $(p, \dot{q}, \dot{m}_0) \leq j(p)$. Thus, it remains to prove that

$$(p, \dot{q}, \dot{m}_0) \Vdash \dot{m}_1 \leq j(\dot{q}).$$

It suffices to show that for every $(p, \dot{q})$ there are densely many conditions $(p^*, \dot{q}^*) \leq (p, \dot{q})$ such that $(p^*, \dot{q}^*, \dot{m}_0) \Vdash \dot{m}_1 \leq j(\dot{q}^*)$.

Fix $(p', \dot{q}') \leq (p, \dot{q})$. Let $(p^*, \dot{q}^*) \leq (p', \dot{q}')$ and $A \in ([\kappa^+]^{<\kappa})^V$ be such that

$$p^* \Vdash A = \operatorname{dom}(\dot{q}^*).$$

[21] Note that it was necessary to add δ, as we typically do not have κ in the generic club of $\mathbb{Q}_{j(\kappa)}^{j(\kappa)^+}$, since $o^{\mathcal{U}}(\kappa) = \kappa^+ + 1$. It was also necessary to add κ since each c_α^g is unbounded in κ. Finally, both κ, δ are measurables in M, so adding them to $\dot{c}_\alpha^g$ results in valid conditions.

Such A can be found because $\mathbb{P}_\kappa$ satisfies the κ-cover property, by Corollary 2.30. Since $|A| < \kappa$, we can assume, by further extending p^*, that for every $\alpha \in A$, there exists $\beta_\alpha < \kappa$ and a $\mathbb{P}_{\beta_\alpha}$-name $\dot{x}_\alpha \in V_\kappa$ such that

$$p^* \restriction \beta_\alpha \Vdash p^* \setminus \beta_\alpha \Vdash \dot{q}^*(\alpha) = \dot{x}_\alpha.$$

Applying j, we obtain that for every $\alpha \in A$,

$$(p^*, \dot{q}^*, \dot{m}_0) \Vdash j(\dot{q}^*)(j(\alpha)) = \dot{x}_\alpha.$$

Since this is true for every $\alpha \in A$, and since $(p^*, \dot{q}^*, \dot{m}_0) \Vdash j[A] = \mathrm{dom}(j(\dot{q}^*))$, the above choice of $\dot{m}_1$ ensures that

$$(p^*, \dot{q}^*, \dot{m}_0) \Vdash \dot{m}_1 \leq j(\dot{q}^*),$$

as desired. □

Now, having constructed a master condition $\dot{m}$, consider the poset

$$\mathbb{R} := \langle j(\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa) \setminus (\kappa + 1), \leq^* \rangle.$$

Claim 5.2.2. *There exists a set $H \in V[G * g]$ such that $H \subseteq \mathbb{R}$ is generic over $M[G * g]$ and $\dot{m}_{G*g} \in H$.*

Proof. Work in $V[G * g]$. The poset $\mathbb{R}$ is λ^+-strategically closed, since it has a dense subset which is λ^+-directed closed (that is, $j(D * \mathbb{D}_\kappa) \setminus (\kappa + 1)$, where D is as in the proof of Claim 5.2.1). It is not hard to verify that there exists in $V[G * g]$ an enumeration $\langle E_\alpha \mid \alpha < \lambda^+ \rangle$ consisting of all dense subsets of $\mathbb{R}$ in $V[G * g]$. Now by a standard argument, construct H by playing below $\dot{m}_{G*g}$ a game of length λ^+ in which one of the players chooses conditions in the dense sets E_α, and the other makes sure that there are lower bounds at limit stages. The moves in the game are the conditions which generate the generic H inside the extension $V[G * g]$. □

We are finally prepared to define a supercompact measure $\mathcal{V}^* \in V[G * g]$ which extends $\mathcal{V}$. For every $X \subseteq (\mathcal{P}_\kappa(\lambda))^{V[G*g]}$, let $X \in \mathcal{V}^*$ if and only if there is a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$-name $\dot{X}$ for X, a condition $(p, \dot{q}) \in G$ and a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$-name $\dot{r}$ with $(p, \dot{q}) \Vdash \dot{r} \in \dot{H}$, such that

$$(p, \dot{q}, \dot{r}) \Vdash j[\check{\lambda}] \in j(\dot{X}).$$

Claim 5.2.3. *$\mathcal{V}^*$ is a fine, normal measure on $(\mathcal{P}_\kappa(\lambda))^{V[G*g]}$ which extends the measure $\mathcal{V}$.*

Proof. It's straightforward to verify that $\mathcal{V}^*$ is well-defined, upward closed and extends $\mathcal{V}$. Those properties imply that $\mathcal{V}^*$ is fine. The fact that $\mathcal{V}^*$ is closed under finite intersections follows from the fact that any pair of conditions in H have a common extension in H. Thus, it remains to prove that $\mathcal{V}^*$ is a κ-complete normal ultrafilter.

Assume that $\langle \dot{X}_i \mid i < \gamma \rangle$ for some $\gamma < \kappa$ is forced by some condition $(p, \dot{q}) \in G * g$ to be a partition of $\dot{\mathcal{P}}_\kappa(\lambda)$. We thus obtain that, over M, $(\dot{p}, \dot{q}, \dot{m})$ forces that $\langle j(\dot{X}_i) \mid i < \gamma \rangle$ is a partition of $j(\dot{\mathcal{P}}_\kappa(\lambda))$. Since $j(\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa) \setminus (\kappa + 1)$ satisfies the Prikry property and $\mathbb{R}$ is λ^+-strategically closed

in $M[G*g]$, the set of conditions in $\mathbb{R}$ which decide the value of i for which $j[\lambda] \in j(\dot{X}_i)$ is $\leq^*$-dense below $\dot{m}$. Since H is $\leq^*$-generic for $\mathbb{R}$,

$$(p, \dot{q}) \Vdash \text{ there are } r \in \dot{H} \text{ and } i < \gamma \text{ such that } r \Vdash j[\lambda] \in j(\dot{X}_i).$$

By extending $(p, \dot{q})$ inside $G * g$, we can assume that $(p, \dot{q})$ decide the value of $\dot{r}, i$. Therefore $(p, \dot{q}, \dot{r}) \Vdash j[\lambda] \in j(\dot{X}_i)$, namely $\dot{X}_i \in \mathcal{V}^*$, as desired.

Finally, let us argue that $\mathcal{V}^*$ is normal. Let $\dot{f}$ be a $\mathbb{P}_\kappa * \mathbb{Q}_\kappa$-name for a regressive function on $\dot{\mathcal{P}}_\kappa(\lambda)$, forced by some condition $(p, \dot{q}) \in G * g$. It thus follows that over M, $(p, \dot{q}, \dot{m})$ forces that $j(\dot{f})(j[\lambda]) \in j[\lambda]$. Since $j(\mathbb{P}_\kappa * \mathbb{Q}_\kappa) \setminus (\kappa + 1)$ satisfies the Prikry property, the set of conditions $r \in \mathbb{R}$ which decide the unique value $\alpha < \lambda$ such that $j(\dot{f})(j[\lambda]) = j(\alpha)$ is a dense subset of $\mathbb{R}$ below $\dot{m}$ which belongs to $M[G*g]$, since $\mathbb{R}$ is λ^+-strategically closed in $M[G*g]$ and $j \restriction \lambda \in M$. Thus,

$$(p, \dot{q}) \Vdash \text{ there are } r \in \dot{H} \text{ and } \alpha < \lambda \text{ such that } r \Vdash j(\dot{f})(j[\lambda]) = j(\alpha).$$

By extending $(p, \dot{q})$ inside $G * g$, we can assume that they decide the value of $\dot{r}, \alpha$. Thus $(p, \dot{q})$ witnesses that $\{x \in (\mathcal{P}_\kappa(\lambda))^{V[G*g]} \mid f(x) = \alpha\} \in \mathcal{V}^*$. □

This proves that κ remains supercompact in $V[G*g]$.

Finally, let us show that κ remains HOD-supercompact in $V[G*g]$ provided $V = \text{gHOD}$ holds:

Claim 5.2.4. *If $V = \text{gHOD}$ holds then $V[G*g] \models \kappa$ is HOD-supercompact.*

Proof of claim. Fix $\lambda > \kappa$ a strong limit cardinal and $\mathcal{V} \in V$ a supercompactness measure on $\mathcal{P}_\kappa(\lambda)$ such that $j_{\mathcal{V}}(\ell)(\kappa) = \lambda$ and $j_{\mathcal{V}}(\mathcal{U}) \restriction \kappa+1 = \mathcal{U}$. Let $\mathcal{V}^* \in V[G*g]$ be a supercompactness measure on $(\mathcal{P}_\kappa(\lambda))^{V[G*g]}$ extending $\mathcal{V}$, constructed as above. We argue that

$$j_{\mathcal{V}^*}(\text{HOD}^{V[G*g]}) \cap (V_\lambda)^{V[G*g]} = (\text{HOD})^{V[G*g]} \cap (V_\lambda)^{V[G*g]}.$$

The forcing $\mathbb{P}_\kappa * \mathbb{Q}_\kappa$ is homogeneous modulo equivalence (see Theorem 4.10). Furthermore, it is ordinal-definable, because $V = \text{gHOD}$. Therefore, by Lemma 2.20, it follows that $\text{HOD}^{V[G*g]} = V$. So, the right hand side is V_λ.

Next, assuming that $j_{\mathcal{V}^*} \colon V[G*g] \to M^*$ is the ultrapower embedding associated with $\mathcal{V}^*$ over $V[G*g]$, let $M = j_{\mathcal{V}^*}(V)$ be an inner model of M^* such that for $G^* = j_{\mathcal{V}^*}(G*g)$, $M^* = M[G^*]$. By elementarity, M satisfies $V = \text{gHOD}$. Again, because $j_{\mathcal{V}^*}(\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa)$ is homogeneous modulo equivalence, the left hand side is equal to $M \cap (V_\lambda)^{V[G*g]}$. Thus, it suffices to argue that $(V_\lambda)^M = V_\lambda$.

Define an embedding $k \colon M_{\mathcal{V}} \to M$ via $k([f]_{\mathcal{V}}) = [f]_{\mathcal{V}^*}$ for every function $f \colon (\mathcal{P}_\kappa(\lambda))^V \to V$ in V. It's not hard to check that k is elementary and

$$j_{\mathcal{V}^*} = k \circ j_{\mathcal{V}}.$$

We argue that $\text{crit}(k) > \lambda$. Let $\mu \leq \lambda$. Then in V,

$$\mu = [x \mapsto \text{otp}(x \cap \mu)]_{\mathcal{V}}$$

and thus

$$k(\mu) = [x \mapsto \mathrm{otp}(x \cap \mu)]_{\mathcal{V}^*} = \mathrm{otp}\left(j_{\mathcal{V}^*}[\lambda] \cap j_{\mathcal{V}^*}(\mu)\right) = \mu.$$

Finally, since $\mathrm{crit}(k) > \lambda$,

$$V_\lambda = (V_\lambda)^{M_{\mathcal{V}}} = k\left((V_\lambda)^{M_{\mathcal{V}}}\right) = (V_\lambda)^M.$$

This yields the desired equalities. □

This concludes the proof of Theorem 5.2. □

Theorem 5.3. *Assume the existence of a supercompact cardinal. Then, there is a cardinal-preserving generic extension where:*

(1) The HOD *hypothesis holds.*
(2) κ is HOD*-supercompact.*
(3) There is a club $C \subseteq \kappa$ consisting of measurable cardinals in HOD*.*

Proof. Without loss of generality, we assume that $V = \mathrm{gHOD}$ and the GCH holds (see Fact 2.4). Working over this model let $\mathcal{U}$ be a coherent sequence of measures as in Lemma 2.35 and force with the associated iteration $\mathbb{P}_{\kappa+1}$. This is an ordinal definable iteration (because $\mathcal{U}$ is OD), and it is almost homogeneous modulo equivalence (by Theorem 4.10). As a result, $\mathrm{HOD}^{V[G*g]} = V$ for every generic $G * g \subseteq \mathbb{P}_{\kappa+1}$. By Claim 5.2.4, κ is HOD-supercompact in $V[G * g]$. In addition, $\mathbb{Q}_\kappa$ shoots a club through the set $\{\alpha < \kappa \mid \alpha \text{ is } V\text{-measurable}\}$, ergo (3) above holds. Finally, (1) above holds in that $\mathbb{P}_{\kappa+1}$ is a set-sized forcing and the HOD hypothesis holds in the ground model. □

Remark 5.4. Due to the flexibility in the construction of $\mathcal{U}$ described in Remark 2.36 we can arrange a situation where κ is HOD-supercompact and, say, $C \subseteq \kappa$ consists of supercompact cardinals in HOD.

A question that has eluded our methods is the following:

Question 2. *Assume large cardinals. Is the conclusion of Theorem 5.3 consistent with "$\theta^{+\mathrm{HOD}} < \theta^+$ for all $\theta \in C$"?*

By results of [GOP24] the above configuration is known to be consistent if κ is just assumed supercompact. The basic issue is the lack of tools to control lifts of normal, fine, θ-complete ultrafilters on $\mathcal{P}_\theta(\theta^+)$ which in turn do not enable the homogeneity arguments given in §4.

6. On successors of singular cardinals in HOD

In this section we want to address an issue raised by Woodin regarding successors of singular cardinals in HOD. In his *Suitable Extenders Model I* paper, on [Woo10, p.299], Woodin claims that if γ is a singular strong limit cardinal and $\gamma^{+\mathrm{HOD}} < \gamma^+$ then either $\mathrm{cf}(\gamma^{+\mathrm{HOD}}) = \omega$ or $\mathrm{cf}(\gamma^{+\mathrm{HOD}}) = \mathrm{cf}(\gamma)$, and immediately after he conjectures that $\mathrm{cf}(\gamma^{+\mathrm{HOD}}) = \omega$ should hold. The same issue appears referenced in Goldberg's [Gol24a, Question 3.8].

In the next theorem we use *Supercompact Radin forcing* to establish a negative answer to this conjecture. We assume familiarity with Supercompact Radin forcing as exposed in [CFG15].

Theorem 6.1. *Suppose that κ is a κ^+-supercompact cardinal and the axiom* $\mathrm{V} = \mathrm{gHOD}$ *holds. Then, there is a generic extension where κ is a strong limit cardinal,* $\mathrm{cf}(\kappa^{+\mathrm{HOD}}) = \omega_1$ *and* $\kappa^{+\mathrm{HOD}} \neq \kappa^+$. *Moreover, in this model* $\mathrm{cf}^{\mathrm{HOD}}(\kappa) = \omega_1$.

Proof. Let $j\colon V \to M$ be an elementary embedding such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \kappa^+$ and $M^{\kappa^+} \subseteq M$. For each $\alpha < \omega_1$ define by induction a sequence $u := \langle u(\alpha) \mid \alpha < \omega_1 \rangle$ as follows: $u(0) := j[\kappa^+]$ and $u(\alpha) := \{X \subseteq V_\kappa \mid u \restriction \alpha \in j(X)\}$. It follows that $u \in \mathcal{U}^{\mathrm{sup}}_\infty$ as defined in [CFG15, Definition 3.6].

Let $\mathbb{R}^{\mathrm{sup}}_u$ be the corresponding supercompact Radin forcing (see [CFG15, §3]) and $G \subseteq \mathbb{R}^{\mathrm{sup}}_u$ generic for it. The analysis of [CFG15, §4] shows that there is a Radin-like extension $V[G_0] \subseteq V[G]$, preserving cardinals of V, such that $\mathrm{HOD}^{V[G]} \subseteq V[G_0]$. Since $V = \mathrm{gHOD}$ holds in V, we have

$$V \subseteq \mathrm{HOD}^{V[G]} \subseteq V[G_0].$$

In $V[G]$, the following hold:

$$\kappa^{+\mathrm{HOD}^{V[G]}} = \kappa^{+V} < \kappa^{++V} = \kappa^{+V[G]}.$$

In addition $\mathrm{cf}^{V[G]}(\kappa^{+\mathrm{HOD}}) = \mathrm{cf}^{V[G]}(\kappa^{+V}) = \mathrm{cf}^{V[G]}(\kappa) = \omega_1$.

We are left with showing that $\mathrm{cf}^{\mathrm{HOD}^{V[G]}}(\kappa) = \omega_1$. Let $C \subseteq \kappa$ be the generic Radin club of order-type ω_1 introduced by G. Note that all points in C are regular in V. We claim that $\mathrm{acc}(C)$ is ordinal definable in $V[G]$. Indeed, we claim that

$$\mathrm{acc}(C) = \{\rho < \kappa \mid V[G] \models \text{“}\mathcal{G}_\kappa \models \text{“}\mathrm{cf}(\rho) = \rho\text{”} \wedge \mathrm{cf}(\rho) < \rho\text{”}\}$$

where $\mathcal{G}$ is a predicate denoting the ground of $V[G]$ and $\mathcal{G}_\kappa$ the predicate for the members of $\mathcal{G}$ of rank $<\kappa$. Since $V = \mathrm{gHOD}$ holds in V the satisfability relation $\models$ is ordinal definable, the right-hand-side is in HOD.

The left-to-right inclusion is obvious as $\mathrm{otp}(C) = \omega_1$. For the converse, suppose that ρ is a V-regular cardinal that changes its cofinality. Let $\alpha < \omega_1$ be the first ordinal such that $\sigma_\alpha \leq \rho < \sigma_{\alpha+1}$. Let $p \in G$ be a condition mentioning measure sequences $u_\alpha, u_{\alpha+1} \in \mathcal{U}^{\mathrm{sup}}_\infty$ such that $\kappa_{u_\alpha} = \sigma_\alpha$ and $\kappa_{u_{\alpha+1}} = \sigma_{\alpha+1}$. By the *Factor Lemma* of Radin forcing, $\mathbb{R}^{\mathrm{sup}}_u/p$ is isomorphic to $\mathbb{R}^{\mathrm{sup}}_{u_\alpha}/p_0 \times \mathbb{R}^{\mathrm{sup}}_u/p_1$ where p_0 is the truncation of p up to the occurence of u_α and p_1 denotes the rest of p. The latter forcing does not change cofinalities below $\sigma_{\alpha+1}$ so it has to be $\mathbb{R}^{\mathrm{sup}}_{u_\alpha}/p_0$ the responsible for changing the cofinality of ρ. However, this poset is σ^+_α-cc so it must be that $\rho = \sigma_\alpha$. If σ_α were to be a successor point of C then it is easy to show (again, using the Factor Lemma) that σ_α does not change its cofinality, ergo ρ is a member of $\mathrm{acc}(C)$, as needed. □

Remark 6.2. Following up [Gol24a, Question 3.8] Goldberg asks if Clause (5) from his [Gol24a, Theorem 3.6(5)] can be ruled out whenever κ is a HOD-singular cardinal; more precisely, he asks if there can be a V-cardinal κ that is HOD-singular and such that $\omega < \mathrm{cf}(\kappa^{+\mathrm{HOD}}) < |\kappa^{+\mathrm{HOD}}|^V$. The above theorem thus answers Goldberg's question in the affirmative.

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